

Universal coding, intrinsic volumes, and metric complexity

Jaouad Mourtada*

March 13, 2023

Abstract

We study sequential probability assignment in the Gaussian setting, where the goal is to predict, or equivalently compress, a sequence of real-valued observations almost as well as the best Gaussian distribution with mean constrained to a given subset of \mathbf{R}^n . First, in the case of a convex constraint set K , we express the hardness of the prediction problem (the minimax regret) in terms of the intrinsic volumes of K ; specifically, it equals the logarithm of the Wills functional from convex geometry. We then establish a comparison inequality for the Wills functional in the general nonconvex case, which underlines the metric nature of this quantity and generalizes the Slepian-Sudakov-Fernique comparison principle for the Gaussian width. Motivated by this inequality, we characterize the exact order of magnitude of the considered functional for a general nonconvex set, in terms of global covering numbers and local Gaussian widths. This implies metric isomorphic estimates for the log-Laplace transform of the intrinsic volume sequence of a convex body. As part of our analysis, we also characterize the minimax redundancy for a general constraint set. We finally relate and contrast our findings with classical asymptotic results in information theory.

Keywords: Sequential prediction, Convex bodies, Intrinsic volumes, Universal coding, Redundancy, Gaussian processes.

Contents

1	Introduction	2
2	Minimax regret, intrinsic volumes and Wills functional	7
2.1	Sequential probability assignment, coding, and minimax regret	7
2.2	Convex bodies and intrinsic volumes	9
2.3	Minimax regret, intrinsic volumes and Wills functional	10
3	Infinite-dimensional version, finiteness and asymptotics	13
3.1	Basic properties and infinite-dimensional extension	13
3.2	Finiteness and connection with Gaussian width	14
3.3	Small and large scale asymptotics	17
4	Comparison theorem and metric characterization	17
4.1	Comparison theorem for the Wills functional	17
4.2	Sharp metric characterization of the minimax regret and Wills functional	20
4.3	Statistical setting: metric characterization of the minimax redundancy	23
4.4	Metric estimates for the intrinsic volume sequence	24

*CREST, ENSAE, Institut polytechnique de Paris; jaouad.mourtada@ensae.fr

5	Interplay with additive structure and noise dependence	26
5.1	Convex combinations	26
5.2	Sum and difference bodies	27
5.3	Dilations and dependence on noise and sample size	29
6	Comparison with classical asymptotics of coding	31
7	Examples	32
7.1	Ellipsoids	32
7.2	The ℓ^1 ball	36
8	Proof of Theorems 4.2 and 4.3	37
8.1	Fixed points and local widths	37
8.2	Upper bound on the redundancy	38
8.3	Lower bound on the redundancy	39
8.4	First upper bound on the regret	42
8.5	Lower bound on the regret	42
8.6	Relating the upper and lower bounds	43
8.7	Generic chaining characterization	43
A	Remaining proofs	45
A.1	Proof of Proposition 2.1	45
A.2	Proof of Proposition 3.4	46
A.3	Simplified form of minimax regret for convex bodies	46
A.4	Short proof of McMullen’s inequality	47

1 Introduction

This work studies universal coding in a Gaussian setting, and relates the hardness of this prediction problem to the geometry of the “model”. This information-theoretic problem will be first connected to classical convex geometry, and then studied from a distinct metric perspective inspired by the theory of Gaussian processes. In an attempt to make this text accessible to readers from various fields, we have included some background material in the first two sections.

Sequential prediction. Sequential probability assignment is a basic problem in information theory, statistics and learning theory, with connections to sequential prediction [Vov98, CBL06], lossless coding and data compression [CT06b, CS04, MF98, Gas18, PW22], minimum description length [Ris85, BRY98, Grü07], as well as aggregation and estimation theory in statistics [Cat04, YB99, Yan00]. Informally, it can be formulated as follows: given a sequence of observations unknown a priori, the goal is to assign a joint probability as large as possible to this sequence.

Specifically, let $y_{1:n} = (y_1, \dots, y_n) \in \mathcal{Y}^n$ be a sequence of observations, taking value in some measurable space \mathcal{Y} , which we aim to predict. Here, the forecast is not “punctual” (that is, consisting of another sequence that should be “close” to the true one in a suitable sense) but rather “probabilistic”: it consists in assigning probabilities to all possible sequences, with the aim of attributing a probability as large as possible to the actual data sequence $y_{1:n}$. Given a reference measure μ on \mathcal{Y} , a prediction then consists of a positive probability density $p : \mathcal{Y}^n \rightarrow \mathbf{R}^+$ with respect to μ^n , whose error on the sequence $y_{1:n}$ is measured by the *logarithmic loss* (or *negative log-likelihood*)

$$\ell(p, y_{1:n}) = -\log p(y_1, \dots, y_n). \tag{1.1}$$

A few comments on the definition and interpretation of the problem are in order.

First, the loss function (1.1) has a precise and classical interpretation in terms of coding and data compression, see e.g. [CT06b, Chapter 5], [Cat04, Chapter 1], [Gas18, Chapter 1] and [PW22, Chapter 13]. Roughly speaking, if \mathcal{Y} is finite and μ is the counting measure, there is an almost exact correspondence between probability densities p on \mathcal{Y}^n on the one hand, and on the other hand codes $\phi : \mathcal{Y}^n \rightarrow \bigcup_{k \geq 1} \{0, 1\}^k$ compressing each sequence $y_{1:n}$ into a string of bits $\phi(y_{1:n})$, with a suitable form of injectivity allowing for decoding. In addition, under this correspondence, the length of the code $\phi(y_{1:n})$ associated to the sequence $y_{1:n}$ coincides (up to an integer rounding error) to the logarithmic loss $-\log_2 p(y_1, \dots, y_n)$ in base 2. (The precise statement underpinning this correspondence, namely *Kraft's inequality*, is included in Section 2.1 and in the aforementioned references. In addition, the correspondence extends from finite sets to continuous spaces \mathcal{Y} by discretization using an increasing sequence of finite partitions.) Because of this, in what follows we will sometimes refer to probability densities p as “codes”, and to the log-likelihood $\ell(p, y_{1:n})$ as the “code-length”.

Second, the problem admits an equivalent “sequential” formulation. Indeed, in the formulation above, a code p on \mathcal{Y}^n must be chosen ahead of time, prior to seeing any observation. A perhaps more appealing formulation is in terms of *sequential prediction* (see [CBL06] for a thorough exposition of this topic). Here, observations are revealed one at a time; at each step, one wishes to issue a probabilistic prediction of the next observation knowing the previous ones. In this context, a *sequential prediction strategy*, or sequential coding scheme, is a sequence (p_1, \dots, p_n) , where p_i is a function associating a density $p_i(\cdot | y_1, \dots, y_{i-1})$ on \mathcal{Y} to each $(y_1, \dots, y_{i-1}) \in \mathcal{Y}^{i-1}$. Now, each such p_i can be seen as a conditional density on y_i given y_1, \dots, y_{i-1} , hence a sequential prediction strategy (p_1, \dots, p_n) amounts to a family of such conditional densities, which in turns amounts to a single joint density p on \mathcal{Y}^n defined by $p(y_1, \dots, y_n) = p_1(y_1)p_2(y_2|y_1) \cdots p_n(y_n|y_1, \dots, y_{n-1})$. In addition, the cumulative loss (total code-length) of the sequential prediction scheme associated to p coincides with the loss (1.1):

$$-\sum_{i=1}^n \log p_i(y_i | y_1, \dots, y_{i-1}) = -\log p(y_1, \dots, y_n) = \ell(p, y_{1:n}).$$

This remarkable tensorization property shows that the previously defined problem is in fact equivalent to sequential prediction.

Statistical model and minimax regret. As formulated, the problem of minimizing the loss (1.1) is ill-defined. Indeed, the normalization constraint—that p must be a probability density on \mathcal{Y}^n —implies that no density can assign a high probability to all sequences. This expresses the fact that no code can efficiently compress all sequences. It follows from this obstruction that one should specify on which “structured” sequences a small loss should be incurred, and quantify what constitutes a good performance.

A common way to achieve this goal is to resort to statistical modeling of the data sequence. Concretely, a *statistical model* \mathcal{P} is a set of probability densities on \mathcal{Y}^n . Perhaps the most classical way to use a model \mathcal{P} is to assume that the sequence of observations $Y_{1:n}$ is random, with density p^* for some (unknown) $p^* \in \mathcal{P}$. In this case, the code with smallest expected code-length is the true density p^* , and if p is another density then $\mathbb{E}[\ell(p, Y_{1:n}) - \ell(p^*, Y_{1:n})] = \text{KL}(p^*, p)$, the Kullback-Leibler divergence between p^* and p [KL51, CT06b] (see Section 2.1 for more details).

An alternative approach (see [CBL06] for a historical account on this perspective) is to avoid making explicit assumptions on the data sequence $y_{1:n} \in \mathcal{Y}^n$, which may be arbitrary, and instead use the statistical model \mathcal{P} as a benchmark. Specifically, define the *regret* (sometimes also called *redundancy*) of a density q (not necessarily belonging to \mathcal{P}) with respect to the model

\mathcal{P} over the sequence $y_{1:n} \in \mathcal{Y}^n$ as

$$\text{Regret}(q, \mathcal{P}, y_{1:n}) = \ell(q, y_{1:n}) - \inf_{p \in \mathcal{P}} \ell(p, y_{1:n}). \quad (1.2)$$

That is, the regret is the difference in code-length between the code q and the best code (a posteriori) in the model \mathcal{P} for the sequence $y_{1:n}$. Since no assumption is made on the sequence $y_{1:n}$, one can evaluate the performance of a distribution q relative to a class \mathcal{P} by the worst-case (maximum) regret over all sequences. This leads to considering the *minimax regret*, which is the best regret guarantee (with respect to \mathcal{P}) one can ensure uniformly over all sequences:

$$\mathbf{R}^*(\mathcal{P}) = \inf_q \sup_{y_{1:n} \in \mathcal{Y}^n} \text{Regret}(q, \mathcal{P}, y_{1:n}), \quad (1.3)$$

where the infimum is taken over all probability densities q on \mathcal{Y}^n . Clearly, the richer the model \mathcal{P} , the harder it is to compete against it, and hence the larger $\mathbf{R}^*(\mathcal{P})$ is. As such, the minimax regret measures the complexity of the model, from the perspective of worst-case sequential prediction.

Gaussian setting. In this work, we consider sequential probability assignment in a Gaussian setting, where the observations y_1, \dots, y_n are real-valued and modeled by a subset of the Gaussian sequence model. This problem can be seen as a coding-theoretic analogue of Gaussian regression. Specifically, for $\theta \in \mathbf{R}^n$, denote by $p_\theta : y \mapsto (2\pi)^{-n/2} e^{-\|y-\theta\|^2/2}$ the density of the Gaussian measure $\mathcal{N}(\theta, I_n)$ with respect to the Lebesgue measure $\mu(dy) = dy$ on \mathbf{R}^n . To any nonempty subset $A \subset \mathbf{R}^n$ (called model or constraint set), one can associate the corresponding subset of the Gaussian model and its corresponding regret:

$$\mathcal{P}_A = \{p_\theta : \theta \in A\}; \quad \mathbf{R}^*(A) = \mathbf{R}^*(\mathcal{P}_A). \quad (1.4)$$

With these definitions at hand, the general question we consider is the following:

How does the complexity $\mathbf{R}^(A)$ of sequential prediction depend on the geometry of the constraint set $A \subset \mathbf{R}^n$?*

Part of the results we will obtain will be specific to the case where A is a convex body, but most of the main results remain in fact valid for general (possibly non-convex) sets. In addition, while our primary focus will be on the worst-case deterministic setting (where hardness is measured by the minimax regret $\mathbf{R}^*(A)$), we will also study the aforementioned statistical setting.

Summary of contributions. In order to convey the principal thread of ideas in this work, we provide a sample of our results. We refer to the corresponding sections for more complete statements and discussion.

The first result states that, in the case where $A = K$ is a convex body in \mathbf{R}^n , the minimax regret $\mathbf{R}^*(K)$ can be expressed in terms of its *intrinsic volumes* $V_j(K)$, $j = 0, \dots, n$. We refer to Section 2.2 for a definition of the intrinsic volumes of a convex body.

Theorem 1.1 (see Theorem 2.1). *If $K \subset \mathbf{R}^n$ is a convex body, the minimax regret $\mathbf{R}^*(K)$ equals*

$$\mathbf{R}^*(K) = \log \left(\sum_{j=0}^n V_j(K/\sqrt{2\pi}) \right).$$

This result connects universal coding to classical convex geometry [Sch13]. In particular, it shows that for convex bodies, the minimax regret can be understood in terms of the volume

behavior of its random projections. This connection is however specific to convex bodies, as it involves quantities such as intrinsic volumes, which are naturally defined on convex sets.

A rather different perspective, which we describe next, extends to general nonconvex sets. An important property of the functional \mathbf{R}^* is that it is a “metric” quantity, which can be understood in terms of distances. This is suggested by the following comparison inequality, which is analogous to the Slepian-Sudakov-Fernique comparison inequality for suprema of Gaussian processes (see Section 4.1 for additional discussion). Here, $\|\cdot\|$ denotes the Euclidean norm.

Theorem 1.2 (see Theorem 4.1). *Let $A, B \subset \mathbf{R}^n$ be two nonempty sets. If there exists a function $\varphi : A \rightarrow B$ such that $\|\varphi(x) - \varphi(y)\| \geq \|x - y\|$ for every $x, y \in A$, then $\mathbf{R}^*(A) \leq \mathbf{R}^*(B)$.*

Theorem 1.2 suggests that the minimax regret $\mathbf{R}^*(A)$ can be understood in terms of distances only, though it does not provide an explicit metric characterization. An explicit characterization can be obtained, relating (up to universal constant factors) the minimax regret $\mathbf{R}^*(A)$ to local Gaussian widths and global covering numbers. Below, for $r > 0$ we denote by $N(A, r)$ the smallest number of closed balls of radius r that cover A , by $B(\theta, r)$ the closed Euclidean ball of radius r centered at $\theta \in \mathbf{R}^n$, and by $w(A) = \mathbb{E} \sup_{\theta \in A} \sum_{i=1}^n \theta_i X_i$ the Gaussian width of A , where X_1, \dots, X_n are independent standard Gaussian variables.

Theorem 1.3 (see Theorem 4.2). *For any subset $A \subset \mathbf{R}^n$, one has*

$$\mathbf{R}^*(A) \asymp \inf_{r \geq 0} \left\{ \sup_{\theta \in A} w(A \cap B(\theta, r)) + \log N(A, r) \right\}.$$

Using Talagrand and Fernique’s Majorizing Measure theorem [Tal87, Tal21], a characterization of $\mathbf{R}^*(A)$ involving only distances can be deduced from Theorem 1.3 (Corollary 4.1). This characterization features a certain truncated version of the generic chaining functional.

Theorems 1.1 and 1.3 provide two different characterizations of the minimax regret over a convex body: the first involving quantities from classical convexity (intrinsic volumes), the second in terms of metric quantities that appear in high-dimensional geometry, probability and statistics. Relating these two characterizations leads to the following result, which provides precise quantitative estimates on the intrinsic volume sequence of a convex body.

Corollary 1.1 (see Corollary 4.2). *For any convex body $K \subset \mathbf{R}^n$ and $\lambda \geq 0$, one has*

$$\log \left(\sum_{j=0}^n V_j(K) \lambda^j \right) \asymp \inf_{r \geq 0} \left\{ \lambda \sup_{\theta \in K} w(K \cap B(\theta, r)) + \log N(K, r) \right\}.$$

Let us mention that the sum of intrinsic volumes of a convex body is known in convex geometry as the Wills functional [Wil73, Had75]. Corollary 1.1 then corresponds to an “isomorphic” characterization of the Wills functional, as discussed in Section 4.4.

Organization. This paper is organized as follows. After some additional background on sequential probability assignment and on intrinsic volumes, Section 2 is devoted to the relation between the two in the case of convex bodies. Section 3 is concerned with a qualitative study of the functional \mathbf{R}^* in the general nonconvex case, including its basic properties, infinite-dimensional version and its finiteness, the link with Gaussian width as well as small and large-scale asymptotics. Then, Section 4 is devoted to the quantitative study of \mathbf{R}^* from a metric perspective; as part of the analysis, we also determine the hardness of sequential prediction in the “statistical” or “average-case” setting. The quantitative characterizations of Section 4 hold up to (universal) constant factors. In Section 5, we complement them by considering some exact structural

properties of the regret, involving its interaction with the additive structure of the space. This allows in particular to establish a form of “strong monotonicity” with respect to the noise level. In Section 6, we relate and contrast our findings to classical asymptotic results from the literature on universal coding and minimum description length. Finally, we consider some concrete examples in Section 7, including the case of ellipsoids.

Related work. Sequential prediction is a topic at the intersection of Game theory, Statistics, Information theory and Learning theory. We refer to the classic textbook [CBL06] for an exposition of the many facets of the subject. We point out some results in sequential prediction which, although not directly related to the problem at hand, also provide precise information in their considered settings. First, in the context of sequential prediction of a binary sequence with absolute error loss (see [CBL06, Chapter 8]), it is shown by Cesa-Bianchi and Lugosi [CBL99] that the minimax regret is equal to the Rademacher average of the class of experts. In addition, a precise description of achievable error bounds for prediction of binary sequences has been obtained by Cover [Cov67]. In the context of prediction of bounded sequences under square loss, Rakhlin and Sridharan [RS14] use certain sequential centered offset Rademacher complexities to sharply control the minimax regret.

A canonical instance of sequential prediction is sequential probability assignment, see [CBL06, Chapter 9] for an introduction to this problem. As discussed previously, it admits a precise interpretation in terms of universal coding [CT06b, Cat04, Gas18, PW22] in information theory. The seminal work of Shtarkov [Sht87] identifies the general properties of the problem, and provides an integral representation of the minimax regret which will be reviewed in Section 2.1. In addition, a classical result of Rissanen [Ris96] provides an asymptotic expansion of the regret for a fixed finite-dimensional model in the limit of large “sample size”, whose relation to the present work we discuss in more detail in Section 6. Finally, Opper and Haussler [OH99] obtain upper bounds on the minimax regret in terms of certain covering numbers of the class; these bounds have been refined in a series of works [CBL01, RS15, GM19, BFR20]. These upper bounds do not apply to the Gaussian setting we consider, but they are nonetheless related to the upper bound in Corollary 4.1. However, they do not come with matching lower bounds.

Intrinsic volumes are equivalent (up to multiplicative constants) to quermassintegrals of convex bodies, themselves a special case of mixed volumes. These quantities are at the core of classical convex geometry, the so-called “Brunn-Minkowski theory”; we refer to the classic book of Schneider [Sch13] for a thorough reference on this topic. The sum of intrinsic volumes, known as *Wills functional*, has been considered by Wills [Wil73] in the context of lattice point enumeration. It has been used in several contexts, including the study of Gaussian processes [Vit96] and in establishing concentration of the sequence of intrinsic volumes [LMN⁺20]. Several notable inequalities on this functional have been obtained by Hadwiger [Had75], McMullen [McM91], and more recently Alonso-Gutiérrez, Hernández Cifre, and Yepes Nicolás [AHY21]. These references are highly relevant to the present work, and we will discuss some of their results in the following sections. We contribute to the study of the Wills functional in two primary ways, first by connecting it to sequential coding, and second by characterizing it (under a suitable normalization) up to universal constant factors.

A singular aspect of Theorem 1.1 is that it features intrinsic volumes, which are uncommon complexity measures in high-dimensional statistics (apart from the first intrinsic volume, namely the Gaussian mean width). A notable exception is the important work [ALM⁺14], which characterizes phase transitions in random convex programs in terms of conic (spherical) intrinsic volumes. Despite the appearance of certain intrinsic volumes in both [ALM⁺14] and the present work, the approaches are different. First, the considered problems differ: [ALM⁺14] deals with linear inverse problems with random measurements (see [CT06a, RV08, DT09, Sto09, CRP⁺12,

FR13] and references therein for more information), while we consider universal coding. Second, [ALM⁺14] features *conic* intrinsic volumes (associated to certain sub-differential cones on the edge of the convex domain), in contrast with the *Euclidean* intrinsic volumes that appear in our context. These quantities are of a different character: conic intrinsic volumes of sub-differential cones are scale-free and reflect the differential and infinitesimal structure of the boundary of the domain, while Euclidean intrinsic volumes are scale-sensitive global complexity measures. This qualitative difference reflects the fact that [ALM⁺14] consider either noiseless exact recovery or noisy recovery in the low-noise limit, while in our context the noise level is fixed.

Notation. Throughout this article, the standard scalar product on the Euclidean space \mathbf{R}^n or on the sequence space $\ell^2 = \{(x_i)_{i \geq 1} \in \mathbf{R}^{\mathbf{N}^*} : \sum_{i \geq 1} x_i^2 < +\infty\}$ is denoted by $\langle \cdot, \cdot \rangle$; we also set $\|x\| = \langle x, x \rangle^{1/2}$ to be the associated ℓ^2 norm. We denote by $B_2^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ the closed unit ℓ^2 ball in \mathbf{R}^n , and let B_2 be either B_2^n or the unit ball in ℓ^2 depending on the ambient space; likewise, for θ in \mathbf{R}^n or in ℓ^2 and for $r \geq 0$, we let $B(\theta, r) = \{x : \|x - \theta\| \leq r\} = \theta + rB_2$ denote the closed ball of radius r centered at θ . The $n \times n$ identity matrix is denoted $I_n \in \mathbf{R}^{n \times n}$. For a measurable subset $A \subset \mathbf{R}^n$, we let $\text{Vol}_n(A)$ denote its volume, namely its n -dimensional Lebesgue measure. In addition, the diameter of $A \subset \mathbf{R}^n$ is $\text{diam}(A) = \sup_{x, y \in A} \|x - y\|$, while the distance of $x \in \mathbf{R}^n$ to A is denoted by $\text{dist}(x, A) = \inf_{\theta \in A} \|\theta - x\|$. Given two sets $A, B \subset \mathbf{R}^n$, their *Minkowski sum* is the set $A + B = \{x + y : x \in A, y \in B\}$; also, $A - B = A + (-B)$. In particular, for $r \geq 0$ the *parallel set* of A with radius r is $\{x \in \mathbf{R}^n : \text{dist}(x, A) \leq r\} = \overline{A} + rB_2$, where \overline{A} is the closure of A .

Additional notation and definitions associated with convex bodies are introduced in Section 2.2 and throughout the text. We use the letters A, B to refer to general (possibly nonconvex) subsets of \mathbf{R}^n or ℓ^2 , and the letters K, L for convex sets (which are often also assumed to be compact). General subsets A, B will typically be assumed nonempty, though this is not always stated explicitly for conciseness. Finally, for two functions $f, g \geq 0$, the notation $f \lesssim g$ means that there exists a constant $C > 0$ such that $f \leq Cg$ everywhere, while $f \asymp g$ means that $f \lesssim g$ and $g \lesssim f$. This notation will only be used when the underlying constants are universal.

2 Minimax regret, intrinsic volumes and Wills functional

In this section, we provide additional background on sequential probability assignment (Section 2.1) and on intrinsic volumes (Section 2.2). The link between the two is then established in Section 2.3.

2.1 Sequential probability assignment, coding, and minimax regret

We start by briefly describing the statement underpinning the connection between sequential probability assignment and coding. Assume that the space \mathcal{Y} of observations is countable and that μ is the counting measure on \mathcal{Y} . A *prefix code* on \mathcal{Y}^n is a function $\phi : \mathcal{Y}^n \rightarrow \bigcup_{k \geq 1} \{0, 1\}^k$ such that for every sequences $y, y' \in \mathcal{Y}^n$, the associated encoding $\phi(y)$ (a string of bits) is not a prefix of $\phi(y')$. (The prefix property is useful to decode concatenated sequences, and relaxing it to the minimal assumption of injectivity of ϕ does not lead to sizeable gains in code-length.) The associated code-length function is $c_\phi(y) = |\phi(y)|$, the length of the string $\phi(y)$. Then, Kraft's inequality (e.g., [CT06b, Theorem 5.2.2 p. 109]) states that a function $c : \mathcal{Y}^n \rightarrow \mathbf{N}^*$ is the code-length function of some prefix code if and only if

$$\sum_{y \in \mathcal{Y}^n} 2^{-c(y)} \leq 1. \quad (2.1)$$

Hence, if ϕ is a prefix code on \mathcal{Y}^n , the positive probability density p_ϕ on \mathcal{Y}^n defined by

$$p_\phi(y) = 2^{-|\phi(y)|} / \left(\sum_{y' \in \mathcal{Y}^n} 2^{-|\phi(y')|} \right)$$

satisfies $-\log_2 p_\phi(y) \leq |\phi(y)|$ for any sequence $y \in \mathcal{Y}^n$. Conversely, if q is a positive probability density on \mathcal{Y}^n , the function $c : \mathcal{Y}^n \rightarrow \mathbf{N}^*$ defined by $c(y) = \lceil -\log_2 q(y) \rceil$ satisfies $\sum_y 2^{-c(y)} \leq \sum_y q(y) = 1$, so by Kraft's inequality there exists a prefix code ϕ_q such that $|\phi_q(y)| = c(y) = \lceil -\log_2 q(y) \rceil \leq -\log_2 q(y) + 1$ for any $y \in \mathcal{Y}^n$. Hence, up to a rounding error (which plays a negligible role for long codes, for instance if the sample size n is large) and a $\log 2$ factor, probability distributions correspond to codes, and the log-loss (1.1) can be interpreted as the associated code-length for the sequence $y_{1:n}$. General observation spaces \mathcal{Y} may be approximated by countable ones using (a sequence of increasingly fine) countable partitions; for instance, the real line $\mathcal{Y} = \mathbf{R}$ may be discretized by the partitions $\{[l/2^k, (l+1)/2^k) : l \in \mathbf{Z}\}$ for $k \geq 1$.

Minimax redundancy and regret. This point now clarified, we come back to the formulation described in the introduction. Let \mathcal{Y} be a measurable observation space endowed with a base measure μ , and \mathcal{P} a *model* on \mathcal{Y}^n , namely a set of probability densities with respect to μ^n .

The two objectives we consider correspond to some form of relative accuracy (in prediction or compression) with respect to the model \mathcal{P} . In the terminology ‘‘universal coding’’, universality refers to the fact that a good accuracy relative to *all* densities $p \in \mathcal{P}$ is sought.

In the statistical setting, the data sequence $Y_{1:n}$ is assumed to be random with density $p \in \mathcal{P}$, where p is unknown. In this case, for any other density q , the difference in average codelength is

$$\mathbb{E}[\ell(q, Y_{1:n}) - \ell(p, Y_{1:n})] = \int_{\mathcal{Y}^n} \log \left(\frac{p(y)}{q(y)} \right) p(y) \mu^n(dy) = \text{KL}(p, q) \geq 0,$$

where $\text{KL}(p, q) = \int p \log(p/q) d\mu^n$ is the *Kullback-Leibler divergence* or *relative entropy* [KL51, CT06b]. This additional expected codelength is sometimes called *redundancy*. It does not depend on the choice of dominating measure μ , but only on the distributions $P = p \cdot \mu$ and $Q = q \cdot \mu$. Recalling that the distribution $p \in \mathcal{P}$ is unknown, a natural measure of the complexity of the problem in the statistical setting is the *minimax redundancy*

$$\text{Red}(\mathcal{P}) = \inf_q \sup_{p \in \mathcal{P}} \mathbb{E}[\ell(q, Y_{1:n}) - \ell(p, Y_{1:n})] = \inf_q \sup_{p \in \mathcal{P}} \text{KL}(p, q), \quad (2.2)$$

where the infimum is with respect to all densities q (not necessarily belonging to \mathcal{P}). This quantity is intrinsic, in the sense that it only depends on the family of distributions $\{p \cdot \mu : p \in \mathcal{P}\}$ on \mathcal{Y}^n and not on the dominating measure μ .

As mentioned in the introduction, an alternative deterministic framework [Sht87, CBL06] is obtained by taking the sequence $y_{1:n} \in \mathcal{Y}^n$ to be arbitrary, and comparing the codelength of a density q to that of the best density in the model \mathcal{P} for any sequence. This leads to the notion of regret (1.2), and to the complexity measure of the problem in the deterministic setting, the *minimax regret* $R^*(\mathcal{P})$ in (1.3), which writes:

$$R^*(\mathcal{P}) = \inf_q \sup_{y \in \mathcal{Y}^n} \log \left(\frac{\sup_{p \in \mathcal{P}} p(y)}{q(y)} \right). \quad (2.3)$$

As shown by Shtarkov [Sht87], this quantity equals the log-integral of the upper envelope of \mathcal{P} :

$$R^*(\mathcal{P}) = \log \left(\int_{\mathcal{Y}^n} \sup_{p \in \mathcal{P}} p(y) \mu^n(dy) \right). \quad (2.4)$$

The reason for this is remarkably simple: due to the constraint that q is a probability density, the infimum is obtained by making the ratio in (2.3) constant over $y \in \mathcal{Y}^n$. Formally, if $C > 0$ satisfies $C > \mathbf{R}^*(\mathcal{P})$, then there exists a density q such that $\sup_{p \in \mathcal{P}} p(y) \leq e^C q(y)$ for any $y \in \mathcal{Y}^n$, so that $\int_{\mathcal{Y}^n} \sup_{p \in \mathcal{P}} p(y) \mu^n(dy) \leq e^C \int_{\mathcal{Y}^n} q = e^C$, which gives the lower bound in (2.4). The upper bound is obtained by taking the density $q(y) = \sup_{p \in \mathcal{P}} p(y) / \int_{\mathcal{Y}^n} \sup_{p \in \mathcal{P}} p(y') \mu^n(dy')$ (known as *normalized maximum likelihood*) whenever the integral in (2.4) is finite.

Remark 2.1 (Measurability issues). When the model \mathcal{P} is countable, the integral in (2.4) is well-defined and this expression for $\mathbf{R}^*(\mathcal{P})$ holds (with the proof above) since the envelope $y \mapsto \sup_{p \in \mathcal{P}} p(y)$ is measurable. In addition, the quantity $\mathbf{R}^*(\mathcal{P})$ does not depend on the choice of measure μ or densities p , but only on the class $\{p \cdot \mu : p \in \mathcal{P}\}$ of probability distributions. When \mathcal{P} is uncountable, the expression (2.4) still holds provided that the integral is an outer integral; however, in this case $\mathbf{R}^*(\mathcal{P})$ depends on the choice of densities p . A better behaved definition for uncountable \mathcal{P} is to define $\mathbf{R}^*(\mathcal{P})$ as the supremum of $\mathbf{R}^*(\mathcal{Q})$ over all countable (or even finite) subsets $\mathcal{Q} \subset \mathcal{P}$, which does not depend on the choice of densities. In the Gaussian case we consider, it equivalently suffices to choose the usual continuous densities, as we do.

It stands to reason that the deterministic setting, which deals with arbitrary data sequences, should be harder than the statistical one. The relationship between minimax redundancy and minimax regret is apparent when writing (inverting the order of suprema in (2.3) for the regret):

$$\begin{aligned} \text{Red}(\mathcal{P}) &= \inf_q \sup_{p \in \mathcal{P}} \left[\int_{\mathcal{Y}^n} \log \left(\frac{p(y)}{q(y)} \right) p(y) \mu(dy) \right]; \\ \mathbf{R}^*(\mathcal{P}) &= \inf_q \sup_{p \in \mathcal{P}} \left[\sup_{y \in \mathcal{Y}^n} \log \left(\frac{p(y)}{q(y)} \right) \right]. \end{aligned}$$

In words, both settings require that the predictive density q be competitive to all densities $p \in \mathcal{P}$ simultaneously; the distinction is in how the difference in codelength between q and p is measured. For the redundancy, the *average* (under p) difference is considered, while for the regret the *worst-case* (maximum) difference is considered.¹ The average being smaller than the supremum, one has

$$\text{Red}(\mathcal{P}) \leq \mathbf{R}^*(\mathcal{P}). \quad (2.5)$$

Most of this work is concerned with the minimax regret, but minimax redundancy will also be considered in Section 4.3.

2.2 Convex bodies and intrinsic volumes

We now recall basic facts and definitions about convexity. A *convex body* is a nonempty convex and compact subset of \mathbf{R}^n for some $n \geq 1$ (following [Sch13], we do not require that this set has nonempty interior). We will need the following quantities associated to a convex body:

Definition 2.1 (Intrinsic volumes). Let $K \subset \mathbf{R}^n$ be a convex body. For $0 \leq j \leq n$, the *j -th intrinsic volume* of K is defined as

$$V_j(K) = \binom{n}{j} \frac{\kappa_n}{\kappa_j \kappa_{n-j}} \mathbb{E}[\text{Vol}_j(P_{\mathbf{V}^j} K)], \quad (2.6)$$

where $\kappa_j = \text{Vol}_j(B_2^j) = \pi^{j/2} / \Gamma(1 + j/2)$, the expectation is with respect to the random subspace \mathbf{V}^j whose distribution is uniform (rotation-invariant) over j -dimensional linear subspaces of \mathbf{R}^n , $P_{\mathbf{V}^j} : \mathbf{R}^n \rightarrow \mathbf{V}^j$ denotes the orthogonal projection on \mathbf{V}^j , and Vol_j refers to the j -dimensional volume (Hausdorff measure) on \mathbf{V}^j induced by its Euclidean structure. In particular, $V_0(K) = 1$.

¹For this reason, the redundancy is sometimes called “average redundancy” while the maximum regret is sometimes called “worst-case redundancy”, though we will not use this terminology to avoid possible confusion.

Intrinsic volumes (as well as the more general *mixed volumes*) play an important role in the theory of convex bodies [Sch13, BZ88] and in stochastic geometry [KR97, SW08] through the so-called *kinematic formula*. The definition (2.6) is known as *Kubota's formula*.

Clearly, the j -th intrinsic volume V_j is homogeneous of degree j , and invariant under affine isometries of \mathbf{R}^n . Some intrinsic volumes amount to familiar quantities: for $K \subset \mathbf{R}^n$ a convex body, $V_n(K)$ is simply the volume $\text{Vol}_n(K)$, while $V_{n-1}(K) = \frac{1}{2}\text{surface}_{n-1}(\partial K)$ is half of the surface area (that is, the $(n-1)$ -dimensional Hausdorff measure) of the boundary ∂K of K . The first intrinsic volume is also particularly important: as shown by Sudakov [Sud76], it corresponds up to a $\sqrt{2\pi}$ factor to the *Gaussian width*; letting $X \sim \mathcal{N}(0, I_n)$, one has

$$V_1(K/\sqrt{2\pi}) = w(K) = \mathbb{E} \sup_{\theta \in K} \langle \theta, X \rangle. \quad (2.7)$$

In fact, the Gaussian representation (2.7) extends to higher-order intrinsic volumes, as shown by Tsirel'son [Tsi86]: for $1 \leq j \leq n$, let G_j be a random $j \times n$ matrix with i.i.d. standard Gaussian entries, seen as a linear map $\mathbf{R}^n \rightarrow \mathbf{R}^j$; then,

$$V_j(K/\sqrt{2\pi}) = \frac{1}{j!} \cdot \mathbb{E} \left[\frac{\text{Vol}_j(G_j K)}{\kappa_j} \right]. \quad (2.8)$$

The representation (2.8) (together with properties of Gaussian vectors) makes it clear that intrinsic volumes do not depend on the ambient dimension, in the sense that $K' = K \times \{0\} \subset \mathbf{R}^{n+m}$ (with $m \geq 1$) satisfies $V_j(K') = V_j(K)$ for all $j = 0, \dots, n$. This property is what motivates (the dependence in n in) the specific choice of normalization constant in the definition (2.6).

While Definition 2.1 has an intuitive geometric appeal, it does not convey what makes intrinsic volumes fundamental quantities associated to a convex body. The canonical nature of these quantities is best appreciated through a striking rigidity theorem of Hadwiger. A *valuation* on convex bodies is a function φ associating a real number to each convex body in \mathbf{R}^n (and 0 to the empty set), satisfying the inclusion-exclusion identity $\varphi(K \cup L) = \varphi(K) + \varphi(L) - \varphi(K \cap L)$ for every convex bodies K, L such that $K \cup L$ is also convex. Consider the linear space of all valuations on convex bodies which are also invariant under affine isometries of \mathbf{R}^n and continuous (with respect to the Hausdorff distance). A priori, there could be infinitely many degrees of freedom for such valuations. However, Hadwiger's theorem (e.g., [Sch13, Theorem 6.4.14 p. 360]) states that this space is in fact finite-dimensional: for any such valuation φ , there exist $c_0, \dots, c_n \in \mathbf{R}$ such that $\varphi(K) = c_0 V_0(K) + \dots + c_n V_n(K)$ for any convex body $K \subset \mathbf{R}^n$.

The reason why intrinsic volumes appear in various contexts (including the present one) is Steiner's classical formula on the volume of parallel sets. Recall that for $r \geq 0$, the *parallel set* of K at radius r is $K + rB_2^n$, the set of points at distance at most r of K . Since the parallel set increases with r (in the sense of inclusion), so does its volume $\text{Vol}_n(K + rB_2^n)$. Steiner's formula (e.g., Equation (4.8) p. 213 in [Sch13]) states that this function of r is in fact a polynomial, whose coefficients are given by the intrinsic volumes: for every $r \geq 0$,

$$\text{Vol}_n(K + rB_2^n) = \sum_{j=0}^n V_{n-j}(K) \kappa_j r^j, \quad \kappa_j = \text{Vol}_j(B_2^j). \quad (2.9)$$

2.3 Minimax regret, intrinsic volumes and Wills functional

With these prerequisites at hand, we turn to our main problem, namely universal coding in the Gaussian setting. Recall from (1.4) that for a nonempty subset $A \subset \mathbf{R}^n$, we let \mathcal{P}_A denote the corresponding subset of the Gaussian model, and $\mathbf{R}^*(A)$ its associated minimax regret.

Before proceeding further, it is worth briefly recalling the way in which the Gaussian model arises and is sometimes formulated. Consider a regression-type setting, where each observation

y_i with $i = 1, \dots, n$ is the *outcome*, or *response*, associated to some *feature* $t_i \in \mathcal{T}$. Here, \mathcal{T} is some abstract set of possible features, which typically consist of several predictive variables. The features (or *design points*) t_1, \dots, t_n are known, but not the associated outcomes $y_1, \dots, y_n \in \mathbf{R}$. Now, given a class \mathcal{F} of functions $\mathcal{T} \rightarrow \mathbf{R}$, let us model

$$Y_i = f(t_i) + \varepsilon_i, \quad i = 1, \dots, n,$$

where $f \in \mathcal{F}$ is some (unknown) regression function in the class, and the errors $\varepsilon_1, \dots, \varepsilon_n$ are assumed to be independent $\mathcal{N}(0, \sigma^2)$ variables. Up to rescaling the observations Y_i , one may reduce to the case $\sigma = 1$; the dependence on the noise level $\sigma > 0$ is discussed in more detail in Section 5.3. This amounts to saying that $Y_{1:n} \sim \mathcal{N}(\theta, I_n)$, where $\theta = (f(t_1), \dots, f(t_n))$. Hence, the corresponding model is simply \mathcal{P}_A , where

$$A = A(\mathcal{F}, t_{1:n}) = \left\{ (f(t_1), \dots, f(t_n)) : f \in \mathcal{F} \right\} \subset \mathbf{R}^n$$

is the set of prediction vectors, which depends on both the class \mathcal{F} and the design points $t_1, \dots, t_n \in \mathcal{T}$. Note that the set A inherits some structural properties of the class \mathcal{F} : for instance, if \mathcal{F} is convex, then so is A . Since the problem only depends on \mathcal{F} and $t_{1:n}$ through the domain $A \subset \mathbf{R}^n$, in what follows we shall only refer to this set.

Our first result expresses the minimax regret in terms of the geometry of the convex domain.

Theorem 2.1 (Minimax regret and intrinsic volumes). *If $K \subset \mathbf{R}^n$ is a convex body, then*

$$\mathbf{R}^*(K) = \log \left(\sum_{j=0}^n V_j(K/\sqrt{2\pi}) \right). \quad (2.10)$$

More generally, for any nonempty subset $A \subset \mathbf{R}^n$,

$$\mathbf{R}^*(A) = \log \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-\text{dist}^2(x,A)/2} dx \right). \quad (2.11)$$

In the case of a convex body K , Theorem 2.1 relates a statistical and information-theoretic quantity, the *minimax regret* measuring the hardness of the prediction problem associated to K , to basic geometric quantities associated to K , namely its *intrinsic volumes*.

Proof. By Shtarkov's integral representation (2.4) of the minimax regret,

$$\mathbf{R}^*(A) = \log \left(\int_{\mathbf{R}^n} \sup_{\theta \in A} \left[\frac{1}{(2\pi)^{n/2}} e^{-\|\theta-x\|^2/2} \right] dx \right) = \log \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-\text{dist}^2(x,A)/2} dx \right)$$

which corresponds to the expression (2.11). In the case where $A = K$ is a convex body, the equality between the right-hand sides of equations (2.11) and (2.10) is due to Hadwiger [Had75]; since the proof is short and follows directly from Steiner's formula, we include it for the reader's convenience. Recall that if (\mathcal{Z}, μ) is a measured space and $f : \mathcal{Z} \rightarrow \mathbf{R}^+$ a measurable function, by Fubini's theorem

$$\int_{\mathcal{Z}} f d\mu = \int_{\mathcal{Z}} \left(\int_0^\infty \mathbf{1}(f \geq t) dt \right) d\mu = \int_0^\infty \mu(\{f \geq t\}) dt.$$

Applying this to $(\mathcal{Z}, \mu) = (\mathbf{R}^n, \text{Vol}_n)$ and $f = e^{-\text{dist}^2(\cdot, A)/2}$, for any nonempty closed $A \subset \mathbf{R}^n$,

$$\begin{aligned} \int_{\mathbf{R}^n} e^{-\text{dist}^2(x,A)/2} dx &= \int_0^\infty \text{Vol}_n \left(\{x \in \mathbf{R}^n : e^{-\text{dist}^2(x,A)/2} \geq t\} \right) dt \\ &= \int_0^\infty \text{Vol}_n \left(\{x \in \mathbf{R}^n : \text{dist}(x, A) \leq r\} \right) r e^{-r^2/2} dr \\ &= \int_0^\infty \text{Vol}_n(A + rB_2^n) r e^{-r^2/2} dr, \end{aligned} \quad (2.12)$$

where the last inequality uses that A is closed (so that $A \cap B(x, r')$ is compact for $x \in \mathbf{R}^n$ and $r' > \text{dist}(x, A)$), allowing to extract a convergent in A sub-sequence of $(\theta_n)_{n \geq 1} \in A^{\mathbf{N}^*}$ such that $\|\theta_n - x\| \rightarrow \text{dist}(x, A)$. Now, assume that $A = K$ is a convex body. Applying Steiner's formula (2.9) in (2.12) gives:

$$\begin{aligned} \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-\text{dist}(x, K)^2/2} dx &= \frac{1}{(2\pi)^{n/2}} \int_0^\infty \left(\sum_{j=0}^n V_j(K) \kappa_{n-j} r^{n-j} \right) r e^{-r^2/2} dr \\ &= \sum_{j=0}^n V_j(K) (2\pi)^{-n/2} \int_0^\infty \text{Vol}_{n-j}(rB_2^{n-j}) r e^{-r^2/2} dr \\ &= \sum_{j=0}^n V_j(K) (2\pi)^{-n/2} (2\pi)^{(n-j)/2} = \sum_{j=0}^n V_j(K/\sqrt{2\pi}), \end{aligned} \quad (2.13)$$

where the penultimate equality (2.13) is obtained by applying (2.12) to $A = \{0\} \subset \mathbf{R}^{n-j}$. \square

For a convex body $K \subset \mathbf{R}^n$, the sum of its intrinsic volumes

$$W(K) = \sum_{j=0}^n V_j(K) \quad (2.14)$$

is called the *Wills functional* [Wil73, Had75]. This quantity was introduced by Wills [Wil73] in the context of lattice point enumeration (that is, the study of the number of points in K with integer coordinates), see the survey [GW93] for more information on this topic. Several relevant results on the Wills functional are obtained in [Wil73, Had75, McM91] and the recent work [AHY21], some of which will be recalled and discussed below. The Wills functional has found applications to Gaussian processes [Vit96] and the study of intrinsic volumes [LMN+20]. As shown by Theorem 2.1, it also has a natural interpretation in terms of universal coding.

Many results in this article beyond Theorem 2.1 will apply to general, not necessarily convex sets. For this reason, it is convenient to extend the definition of the Wills functional to general subsets $A \subset \mathbf{R}^n$ by

$$W(A) = \int_{\mathbf{R}^n} e^{-\pi \text{dist}^2(x, A)} dx. \quad (2.15)$$

Several results in the following sections support the idea that this definition is the most appropriate extension of the Wills functional to nonconvex sets, a first indication of this being the fact that the characterization (2.11) of the minimax regret writes:

$$\mathbf{R}^*(A) = \log W(A/\sqrt{2\pi}) \quad (2.16)$$

for any nonempty domain $A \subset \mathbf{R}^n$. While individual intrinsic volumes (and Steiner's formula) are specific to convex bodies, their sum—the Wills functional—may be thought as defined for general subsets $A \subset \mathbf{R}^n$ via (2.15), since many of its fundamental properties extend to the nonconvex case.

Before this, in the remainder of this section we comment on the case of convex bodies. First, a fundamental property of the sequence of intrinsic volumes of a convex body $K \subset \mathbf{R}^n$ is its *Poisson-log-concavity*, which states that for $1 \leq j \leq n-1$,

$$(j+1)V_{j+1}(K)V_{j-1}(K) \leq jV_j(K)^2. \quad (2.17)$$

This inequality, a consequence of the general Alexandrov-Fenchel inequality between mixed volumes [Sch13, Section 7.3]; [BZ88, §20], was obtained independently by Chevet [Che76] and McMullen [McM91].

A consequence of inequality (2.17) is that the sequence of intrinsic volumes exhibits strong concentration (that is, decay before and after it reaches its maximal value), as shown in [LMN⁺20, AMM22]. This phenomenon suggests that in the expression (2.10) of the minimax regret, one may replace the sum of intrinsic volumes by the largest one without affecting the order of magnitude of this quantity. The following result states that this is indeed the case, and that one may additionally restrict to indices on a geometric grid.

Proposition 2.1. *If $K \subset \mathbf{R}^n$ is a convex body, and if $w(K) = V_1(K/\sqrt{2\pi}) \geq 2$, then*

$$\log \left(\max_{1 \leq j \leq n} V_j(K/\sqrt{2\pi}) \right) \leq \mathbf{R}^*(K) \leq 8 \log \left(\max_{k \geq 0} V_{2^k}(K/\sqrt{2\pi}) \right). \quad (2.18)$$

A proof of Proposition 2.1 is provided in Appendix A.1. This proof does not rely on the concentration results in [LMN⁺20, AMM22], but instead proceeds directly from inequality (2.17).

The restriction to $w(K) \gtrsim 1$ in Proposition 2.1 is necessary, since for $w(K) < 1$ the right-hand side is negative. The regime $w(K) \gg 1$ is arguably the most relevant in the context of sequential probability assignment, especially in high dimension; in any case, for $w(K) \lesssim 1$ one simply has $\mathbf{R}^*(K) \asymp w(K)$ by Proposition 3.2 below.

3 Infinite-dimensional version, finiteness and asymptotics

In this section, we consider the infinite-dimensional version of the Wills functional and minimax regret, and describe some of its qualitative properties such as finiteness and asymptotics. Some first quantitative bounds are discussed along the way, but we defer the study of sharp quantitative estimates to Section 4.

3.1 Basic properties and infinite-dimensional extension

We start with some basic properties of the Wills functional and minimax regret, which follow from the definition (2.11) and will be used repeatedly.

Proposition 3.1. *The minimax regret satisfies the following properties:*

1. *For any subsets $A \subset B \subset \mathbf{R}^n$, one has $\mathbf{R}^*(A) \leq \mathbf{R}^*(B)$.*
2. *For any $A \subset \mathbf{R}^n$, one has $\mathbf{R}^*(\bar{A}) = \mathbf{R}^*(A)$, where \bar{A} is the closure of A .*
3. *If $(A_k)_{k \geq 1}$ is a sequence of subsets of \mathbf{R}^n such that $A_k \subset A_{k+1}$ for any $k \geq 1$ and if $A = \bigcup_{k \geq 1} A_k$, then $\mathbf{R}^*(A) = \lim_{k \rightarrow \infty} \mathbf{R}^*(A_k)$.*
4. *If $A \subset \mathbf{R}^m$ and $B \subset \mathbf{R}^n$ (so that $A \times B \subset \mathbf{R}^{m+n}$), then $\mathbf{R}^*(A \times B) = \mathbf{R}^*(A) + \mathbf{R}^*(B)$.*
5. *If $\iota : \mathbf{R}^n \rightarrow \mathbf{R}^m$ (with $m \geq n$) is an affine isometry, in the sense that ι is affine and $\|\iota(x) - \iota(y)\| = \|x - y\|$ for any $x, y \in \mathbf{R}^n$, then $\mathbf{R}^*(\iota(A)) = \mathbf{R}^*(A)$ for any $A \subset \mathbf{R}^n$.*

Proof. The first property comes from the fact that $\text{dist}(\cdot, A) \geq \text{dist}(\cdot, B)$; the second from the identity $\text{dist}(\cdot, \bar{A}) = \text{dist}(\cdot, A)$; the third from the fact that the increasing sequence of positive functions $e^{-\text{dist}^2(\cdot, A_k)/2}$ (for $k \geq 1$) converges towards $e^{-\text{dist}^2(\cdot, A)/2}$, which concludes by monotone convergence; and the fourth from the identity $\text{dist}^2((x, y), A \times B) = \text{dist}^2(x, A) + \text{dist}^2(y, B)$ for $(x, y) \in \mathbf{R}^m \times \mathbf{R}^n$ (followed by Fubini's theorem).

To prove the fifth property, we proceed in two steps. First, assume that $m = n$; in this case, $\iota(x) = Ux + v$ for every $x \in \mathbf{R}^n$, for some orthogonal matrix $U \in \mathbf{O}(n)$ and $v \in \mathbf{R}^n$. Then $\mathbf{R}^*(\iota(A)) = \mathbf{R}^*(A)$ comes from the fact that ι is a bijection preserving the Lebesgue measure and

that $\text{dist}(\iota(x), \iota(A)) = \text{dist}(x, A)$ for any $x \in \mathbf{R}^n$. Consider now the general case $m \geq n$. Let $\tilde{\iota}$ be an affine isometry of \mathbf{R}^m such that $\tilde{\iota}(\iota(x)) = (x, 0) \in \mathbf{R}^n \times \mathbf{R}^{m-n} \simeq \mathbf{R}^m$ for any $x \in \mathbf{R}^n$. By the case $m = n$ and the fourth property, one has $\mathbf{R}^*(\iota(A)) = \mathbf{R}^*(\tilde{\iota}(\iota(A))) = \mathbf{R}^*(A \times \{0\}) = \mathbf{R}^*(A) + \mathbf{R}^*(\{0\}) = \mathbf{R}^*(A)$. \square

The properties in Proposition 3.1 allow to extend the definition of the minimax regret functional \mathbf{R}^* (or equivalently of the Wills functional) to subsets of the infinite-dimensional sequence space ℓ^2 in the usual way: for any nonempty subset $A \subset \ell^2$, let

$$\mathbf{R}^*(A) = \sup \{ \mathbf{R}^*(B) : B \subset A \text{ finite dimensional} \} \in \mathbf{R}^+ \cup \{+\infty\} \quad (3.1)$$

where a finite-dimensional set B is a set whose affine span is finite-dimensional, and for such a set the Wills functional/minimax regret is defined through an affine isometry between this span and \mathbf{R}^n for some n (by Proposition 3.1, this value does not depend on the choice of the affine isometry). It follows from Proposition 3.1 (specifically, the monotonicity property) that this definition coincides with the previous one for a finite-dimensional set. Note that in the definition (3.1), “finite-dimensional” could be replaced by “finite” due to monotone convergence and stability under closure properties of \mathbf{R}^* .

The infinite-dimensional extension of the Wills functional is similar to that of another dimension-free quantity, namely the Gaussian width, defined for $A \subset \ell^2$ as

$$w(A) = \sup \{ w(B) : B \subset A \text{ finite} \}.$$

Likewise, intrinsic volumes of a closed convex set $K \subset \ell^2$ are defined by taking the supremum over finite-dimensional convex bodies $L \subset K$, and the identity $V_1(K/\sqrt{2\pi}) = w(K)$ still holds.

3.2 Finiteness and connection with Gaussian width

Perhaps the most basic question one may ask about the minimax regret is: for which sets is this functional finite? The purpose of this section is to address this question. As we will see, finiteness of the minimax regret is a more interesting property in infinite dimension than in finite dimension, since it depends on both the “scale” and “dimensionality” of the domain.

A starting point is the following inequality, due to McMullen [McM91, Theorem 2]: for every convex body $K \subset \mathbf{R}^n$,

$$W(K) \leq e^{V_1(K)}. \quad (3.2)$$

This inequality is a consequence of the Poisson-log-concavity property (2.17), which implies (by induction) that $V_j(K) \leq V_1(K)^j/j!$ for every $j \geq 0$; summing over $j \geq 0$ yields (3.2). (An alternative proof is provided in [AHY21], where it is deduced from a functional form of the Blaschke-Santaló inequality from [AKM04, KM05]; in Section A.4, we provide another short proof of this inequality, which essentially only uses the Prékopa-Leindler inequality.) Inequality (3.2) is of isoperimetric nature: it stems from the fact that, for any $n \geq 1$, among all n -dimensional convex bodies with fixed first intrinsic volume, the higher-order intrinsic volumes (and thus also the Wills functional (2.14)) are maximized by a multiple of the unit ball B_2^n —a consequence of the Alexandrov-Fenchel inequality. Letting $n \rightarrow \infty$ and evaluating the limiting Wills functional for these rescaled ℓ^2 balls of increasing dimension leads to the dimension-free inequality (3.2).

While the isoperimetric inequality (3.2) is not sharp in many situations of interest, it plays an important role in the quantitative study of the minimax regret and Wills functional carried out in Section 4 (as well as the qualitative study in the present section).

The Gaussian width is defined in terms of the expectation of a function of a Gaussian vector (2.7), and intrinsic volumes admit a similar representation (2.8). The Wills functional also admits a convenient Gaussian representation [Vit96]: if $A \subset \mathbf{R}^n$, letting $X \sim \mathcal{N}(0, I_n)$,

$$\begin{aligned} W(A/\sqrt{2\pi}) &= (2\pi)^{-n/2} \int_{\mathbf{R}^n} e^{-\text{dist}^2(x,A)/2} dx \\ &= (2\pi)^{-n/2} \int_{\mathbf{R}^n} \exp\left(\frac{\|x\|^2}{2} - \inf_{\theta \in A} \frac{\|\theta - x\|^2}{2}\right) e^{-\|x\|^2/2} dx \\ &= \mathbb{E} \exp\left\{\sup_{\theta \in A} \left[\langle \theta, X \rangle - \frac{\|\theta\|^2}{2}\right]\right\}. \end{aligned} \quad (3.3)$$

A remarkable consequence of this representation is the following inequality, due to Alonso-Gutiérrez, Hernández Cifre, and Yepes Nicolás [AHY21, Theorem 1.2]: if $A \subset B(\theta, r)$ for some $\theta \in \mathbf{R}^n$ and $r > 0$, then

$$\log W(A/\sqrt{2\pi}) \geq w(A) - \frac{r^2}{2}. \quad (3.4)$$

(In [AHY21] this inequality is stated for convex bodies, but the proof only uses the integral representation (2.15) and does not rely on convexity.) Indeed, up to translating A (see Proposition 3.1), we may assume that $\theta = 0$ so $A \subset rB_2^n$. Plugging this in (3.3) and applying Jensen's inequality gives

$$W(A/\sqrt{2\pi}) \geq \mathbb{E} \exp\left\{\sup_{\theta \in A} \left[\langle \theta, X \rangle - \frac{r^2}{2}\right]\right\} \geq \exp\left\{\mathbb{E} \sup_{\theta \in A} \langle \theta, X \rangle - \frac{r^2}{2}\right\} = \exp\left\{w(A) - \frac{r^2}{2}\right\}.$$

The lower bound (3.4) can be seen as a partial reversal of the upper bound (3.2). Since one can always take $r = \text{diam}(A)$ in (3.4), this lower bound implies that $\mathbf{R}^*(A)$ is of order $w(A)$ when $\text{diam}^2(A) \ll w(A)$.

Similarly to the upper bound (3.2), the lower bound (3.4) is typically not sharp. It is fact negative (and thus trivial) in many regimes of interest, in particular in the ‘‘large-scale’’ (or ‘‘low noise’’, ‘‘large sample size’’—the meaning of these terms being clarified in Section 5.3) regime. Nevertheless, once suitably strengthened, this lower bound will play an important role in Section 4.

We now state the main result of this section, namely a characterization of sets $A \subset \ell^2$ for which the Wills functional and minimax regret are finite.

Proposition 3.2. *If $K \subset \ell^2$ is closed and convex, then*

$$\log(1 + w(K)) \leq \mathbf{R}^*(K) \leq w(K). \quad (3.5)$$

In particular, $\mathbf{R}^(K) < \infty$ if and only if $w(K) < \infty$.*

More generally, for any $A \subset \ell^2$, $\mathbf{R}^(A) < \infty$ if and only if $w(A) < \infty$. In addition, $\mathbf{R}^*(A) \leq w(A)$. However, $\mathbf{R}^*(A)$ cannot be lower-bounded in terms of $w(A)$ in the nonconvex case: for any $t > 0$, there exists $A \subset \ell^2$ such that $w(A) \geq t$ but $\mathbf{R}^*(A) \leq 1$.*

The proof of Proposition 3.2 is provided below. The statements in the convex case follow from Theorem 2.1 and the results above, the main point left to prove being that for a nonconvex set $A \subset \ell^2$, if $\mathbf{R}^*(A) < \infty$ then $w(A) < \infty$. The proof actually provides an explicit lower bound, but this quantitative lower bound is highly suboptimal. Much more precise lower bounds will be obtained in Section 4. We have nonetheless included this lower bound since it is simpler than those of Section 4 and suffices to characterize finiteness.

Proposition 3.2 characterizes finiteness of the minimax regret for general sets $A \subset \ell^2$, a necessary and sufficient condition being finiteness of the Gaussian width. (Sets $A \subset \ell^2$ such

that $w(A) < \infty$ are sometimes called ‘‘Gaussian bounded sets’’ in the literature.) However, the quantitative aspect of this qualitative characterization is rather weak: in the nonconvex case, $w(A)$ provides an upper bound on $\mathbf{R}^*(A)$, but no lower bound in terms of $w(A)$ holds. In the convex case, the minimax regret can be lower-bounded in terms of the Gaussian width, but there is an exponential gap between the upper and lower bounds in (3.5).

On this last point, we note that both the upper and lower bounds in (3.5) are best possible in terms of the Gaussian width for convex sets. Indeed, for any $\theta \in \mathbf{R}^+$, the segment $K = [0, \sqrt{2\pi}\theta]$ satisfies $w(K) = V_1([0, \theta]) = \theta$ and $\mathbf{R}^*(K) = \log(1 + \theta)$ (since $V_j(K) = 0$ for $j \geq 2$). On the other end, the upper bound is also optimal: for any $\theta \geq 0$, it follows from (7.3) that $w(\frac{\theta}{\sqrt{n}}B_2^n) \rightarrow \theta$ as $n \rightarrow \infty$, while $\text{diam}(\frac{\theta}{\sqrt{n}}B_2^n) = \frac{\theta^2}{n} \rightarrow 0$, so combining (3.5) and (3.4) gives

$$\lim_{n \rightarrow \infty} \mathbf{R}^*\left(\frac{\theta}{\sqrt{n}}B_2^n\right) = \theta. \quad (3.6)$$

It follows from this exponential gap that the Gaussian width does not suffice to characterize the order of magnitude of \mathbf{R}^* . In the case of convex sets, one must consider the higher-order intrinsic volumes to obtain the correct scaling of the minimax regret. In Section 4, we will see an alternative characterization in terms of different complexity parameters for general sets.

Proof of Proposition 3.2. The upper bound in (3.5) corresponds to McMullen’s inequality (3.2) (since $\mathbf{R}^*(K) = \log W(K/\sqrt{2\pi})$ and $w(K) = V_1(K/\sqrt{2\pi})$), while the lower bound follows from Theorem 2.1 since $V_j(K/\sqrt{2\pi}) \geq 0$ for $j \geq 2$.

If $A \subset \ell^2$, let us show that $\mathbf{R}^*(A) \leq w(A)$. By monotone convergence (Proposition 3.1) and an approximation argument, one may assume that A is a compact subset of \mathbf{R}^n , so that $K = \text{conv}(A)$ is a convex body. By monotonicity and (3.5), $\mathbf{R}^*(A) \leq \mathbf{R}^*(K) \leq w(K) = w(A)$. In particular, if $w(A) < \infty$ then $\mathbf{R}^*(A) < \infty$.

Conversely, let us show that if $A \subset \ell^2$ is such that $\mathbf{R}^*(A) < \infty$, then $w(A) < \infty$. For this, it suffices to show that A is bounded, since in this case applying (3.4) with $r = \text{diam}(A)$ yields $w(A) \leq \mathbf{R}^*(A) + \text{diam}^2(A)/2 < \infty$. We proceed by contraposition, proving that if $A \subset \ell^2$ is unbounded then $\mathbf{R}^*(A) = +\infty$.

For any $N \geq 2$, one can find elements $\theta_1, \dots, \theta_N \in A$ such that $\|\theta_i - \theta_j\| \geq 5\sqrt{N}$ for $i \neq j$ (start with any $\theta_1 \in A$, and given $\theta_1, \dots, \theta_{i-1}$ pick any $\theta_i \in A \setminus \bigcup_{1 \leq j < i} B(\theta_j, 5\sqrt{N})$, which exists since A is unbounded). Let $W \subset \ell^2$ be the affine span of $A_N = \{\theta_1, \dots, \theta_N\}$, and $n \leq N$ its dimension. Let $\iota : W \rightarrow \mathbf{R}^n$ be an affine isometry and $A'_N = \{\vartheta_1, \dots, \vartheta_N\}$ with $\vartheta_i = \iota(\theta_i)$; by Proposition 3.1, $\mathbf{R}^*(A) \geq \mathbf{R}^*(A_N) = \mathbf{R}^*(A'_N)$. Now, for $1 \leq i \leq N$, let $B_i = B(\vartheta_i, 2\sqrt{N}) \subset \mathbf{R}^n$; since $\|\vartheta_i - \vartheta_j\| = \|\theta_i - \theta_j\| \geq 5\sqrt{N}$ for $i \neq j$, the balls $(B_i)_{1 \leq i \leq N}$ are pairwise disjoint. In addition, letting p_{ϑ} be the density of the measure $\mathcal{N}(\vartheta, I_n)$ and $X \sim \mathcal{N}(0, I_n)$, one has for $i = 1, \dots, n$,

$$1 - \int_{B_i} p_{\vartheta_i} = \mathbb{P}((\vartheta_i + X) \notin B(\vartheta_i, 2\sqrt{N})) = \mathbb{P}(\|X\| > 2\sqrt{N}) \leq \frac{\mathbb{E}\|X\|^2}{4N} = \frac{n}{4N} \leq \frac{1}{4}.$$

One can therefore write:

$$\mathbf{R}^*(A'_N) = \log\left(\int_{\mathbf{R}^n} \max_{1 \leq i \leq N} p_{\vartheta_i}\right) \geq \log\left(\sum_{i=1}^N \int_{B_i} p_{\vartheta_i}\right) \geq \log(3N/4).$$

Hence $\mathbf{R}^*(A) \geq \log(3N/4)$ for any $N \geq 2$ and thus $\mathbf{R}^*(A) = +\infty$.

To see that $\mathbf{R}^*(A)$ cannot be lower-bounded in terms of $w(A)$, consider for any $t > 0$ the set $A = \{0, t\} \subset \mathbf{R}$. One has $w(A) = t/\sqrt{2\pi}$, but $\mathbf{R}^*(A) \leq \log 2 \leq 1$ since A has at most two elements (see (8.7)). \square

3.3 Small and large scale asymptotics

Having characterized finiteness of the minimax regret and Wills functional, in this section we conclude the qualitative study of these functionals by discussing small and large scale asymptotics; namely, the asymptotic behavior of $R^*(tA)$ as $t \rightarrow 0$ or $t \rightarrow \infty$.

For a convex body $K \subset \mathbf{R}^n$, the asymptotic behavior of $R^*(tK)$ is readily understood from the expression (2.10), which implies that

$$R^*(tK) \sim tw(K) \quad \text{as } t \rightarrow 0; \quad R^*(tK) = n \log t + \log \text{Vol}_n(K/\sqrt{2\pi}) + o(1) \quad \text{as } t \rightarrow \infty,$$

the latter being true assuming (up to restricting to a subspace) that the affine span of K is \mathbf{R}^n , or equivalently $\text{Vol}_n(K) > 0$. This asymptotic behavior subsists in the nonconvex case, at both small and large scales.

Proposition 3.3. *For any $A \subset \ell^2$ such that $R^*(A) < +\infty$, one has $R^*(tA) \sim t \cdot w(A)$ as $t \rightarrow 0$. In addition, if $\text{diam}^2(A) \leq w(A)$ (which holds if $\text{diam}(A) \leq 1/\sqrt{2\pi}$), then $R^*(A) \geq w(A)/2$.*

Proof. Recall that $R^*(A) < \infty$ if and only if $w(A) < \infty$, and that $R^*(tA) \leq tw(A)$ for any $t > 0$. In this case, (3.4) gives $R^*(tA) \geq tw(A) - t^2 \text{diam}^2(A)/2 \sim tw(A)$ as $t \rightarrow 0$. The second inequality also follows from (3.4), and the fact that $\text{diam}(A) \leq 1/\sqrt{2\pi}$ implies $\text{diam}^2(A) \leq w(A)$ follows from the inequality $w(A) \geq \text{diam}(A)/\sqrt{2\pi}$ (verified by reducing to two-point sets). \square

We now turn to large-scale asymptotics, in finite dimension as in the convex case. (The large-scale asymptotic behavior can be more general in the infinite-dimensional case. We do not expand on this since Section 4 deals with the non-asymptotic order of magnitude, which gives more information than the asymptotic behavior.)

Proposition 3.4. *For any compact set $A \subset \mathbf{R}^n$, one has $\lim_{t \rightarrow \infty} W(tA)/t^n = \text{Vol}_n(A)$. In particular, if $\text{Vol}_n(A) > 0$, then $R^*(tA) = n \log t + \log \text{Vol}_n(A/\sqrt{2\pi}) + o(1)$ as $t \rightarrow \infty$.*

The proof of Proposition 3.4 is provided in Appendix A.2. The assumption that $A \subset \mathbf{R}^n$ is compact is not restrictive, since by Proposition 3.2 (in the finite-dimensional case) $W(A)$ is finite if and only if A is bounded, and by Proposition 3.1 one has $W(A) = W(\overline{A})$, where \overline{A} is closed bounded in \mathbf{R}^n and therefore compact.

4 Comparison theorem and metric characterization

We now come back to the quantitative study of the minimax regret and Wills functional. Theorem 2.1 provides an exact formula in terms of intrinsic volumes in the convex case. In this section, we obtain a characterization in terms of different geometric complexity measures, which holds in the general nonconvex case and also provides new information on intrinsic volumes.

4.1 Comparison theorem for the Wills functional

An important feature of the Wills functional and minimax regret, which is not immediately apparent from the definition (2.15), is that they are in fact “metric” quantities. The metric character of the Wills functional means that, if a set $A \subset \ell^2$ is “larger” than another set B in terms of distances, then $W(A) \geq W(B)$. This is expressed by the following comparison principle:

Theorem 4.1 (Comparison inequality). *Let A, B be two nonempty subsets of ℓ^2 such that there exists a map $\varphi : A \rightarrow B$ such that $\varphi(A) = B$ and $\|\varphi(\theta) - \varphi(\theta')\| \leq \|\theta - \theta'\|$ for all $\theta, \theta' \in A$. Then, one has $W(A) \geq W(B)$ and $R^*(A) \geq R^*(B)$.*

Theorem 4.1 is an analogue of a fundamental result in the theory of Gaussian processes, namely the Sudakov-Fernique comparison inequality [Sud76, Fer75] (which is closely related to Slepian's lemma [Sle62]), which states that under the assumptions of Theorem 4.1 one has $w(A) \geq w(B)$. In fact, Theorem 4.1 generalizes the Slepian-Sudakov-Fernique comparison inequality, which is recovered as follows: for any $t > 0$, the map $\varphi_t(x) = t\varphi(x/t)$ contracts tA into tB , so by Theorem 4.1 one has $\mathbf{R}^*(tA) \geq \mathbf{R}^*(tB)$. Now, by Proposition 3.3 as $t \rightarrow 0$ one has $\mathbf{R}^*(tA) \sim t \cdot w(A)$ and $\mathbf{R}^*(tB) \sim t \cdot w(B)$, so that $w(A) \geq w(B)$.

The proof of Theorem 4.1 relies on classical arguments in the proof of comparison inequalities, namely Gaussian representation, Gaussian interpolation and integration by parts, and smoothing (see [Kah86], [Tal11, §1.3], [Ver18, §7.2] and especially [Cha05]). Interestingly, while the Gaussian representation (3.3) of the Wills functional features an additional variance correction (a second source of dependence on the covariance of the underlying Gaussian process), the proof involves simplifications analogous to those that appear in the proof of Slepian's inequality.

Proof. The inequalities for the Wills functional and minimax regret are equivalent by Theorem 2.1, since a contraction $\varphi : A \rightarrow B$ can be rescaled into a contraction $A/\sqrt{2\pi} \rightarrow B/\sqrt{2\pi}$.

In addition, by an approximation argument, it suffices to prove Theorem 4.1 in the case where A, B are finite. Specifically, since $B \subset \ell^2$ which is separable, one may find a sequence $(y_n)_{n \geq 1} \in B^{\mathbf{N}^*}$ which is dense in B . Hence, letting $B_n = \{y_i : 1 \leq i \leq n\}$ for $n \geq 1$, the sequence $(B_n)_{n \geq 1}$ is increasing and such that $\bigcup_{n \geq 1} B_n$ is dense in B , so that (by Proposition 3.1) $W(B_n) \rightarrow W(B)$. Now since $B = \varphi(A)$, one can write $y_n = \varphi(x_n)$ for some $x_n \in A$ for all n , and letting $A_n = \{x_i : 1 \leq i \leq n\}$ one has $W(A) \geq W(A_n)$ since $A_n \subset A$. It therefore suffices to prove that $W(A_n) \geq W(B_n)$ for all n , noting that $B_n = \varphi(A_n)$. Finally, since $A_n, B_n \subset \ell^2$ have at most n elements, by linearly embedding their respective linear spans in \mathbf{R}^n and using Proposition 3.1, one may also assume that A, B are finite subsets of \mathbf{R}^n .

We now turn to the main proof. Let $A, B \subset \mathbf{R}^n$ be finite sets as in Theorem 4.1, and write $A = \{\theta_i : 1 \leq i \leq N\}$ for some N . By the Gaussian representation (3.3), one has

$$W(A/\sqrt{2\pi}) = \mathbb{E} \exp \left\{ \max_{1 \leq i \leq N} \left[\langle \theta_i, Z \rangle - \frac{\|\theta_i\|^2}{2} \right] \right\}$$

where $Z \sim \mathcal{N}(0, I_n)$. Now, $(\langle \theta_i, Z \rangle)_{1 \leq i \leq N}$ is a centered Gaussian vector, with covariance $\Sigma^A \in \mathbf{R}^{N \times N}$ whose (i, j) -entry is given by $\Sigma_{ij}^A = \mathbb{E} \langle \theta_i, Z \rangle \langle \theta_j, Z \rangle = \langle \theta_i, \theta_j \rangle$ —the Gram matrix of the family $(\theta_i)_{1 \leq i \leq N}$. In addition, $\|\theta_i\|^2 = \Sigma_{ii}^A$ for any $i \leq N$. It follows that

$$W(A/\sqrt{2\pi}) = \mathbb{E} \exp \left\{ \max_{1 \leq i \leq N} [X_i^A - \Sigma_{ii}^A/2] \right\}$$

where $X^A = (X_i^A)_{1 \leq i \leq N} \sim \mathcal{N}(0, \Sigma^A)$. Likewise and since $B = \varphi(A)$, letting $\Sigma^B = (\Sigma_{ij}^B)_{1 \leq i, j \leq N}$ with $\Sigma_{ij}^B = \langle \varphi(\theta_i), \varphi(\theta_j) \rangle$ and $X^B \sim \mathcal{N}(0, \Sigma^B)$ (independent of X^A), one has

$$W(B/\sqrt{2\pi}) = \mathbb{E} \exp \left\{ \max_{1 \leq i \leq N} [X_i^B - \Sigma_{ii}^B/2] \right\}.$$

For every $\beta > 0$, define a function $f_\beta : \mathbf{R}^n \times \mathbf{R}^{n \times n} \rightarrow \mathbf{R}$ by

$$f_\beta(x, \Sigma) = \left\{ \sum_{i=1}^N e^{\beta[x_i - \Sigma_{ii}/2]} \right\}^{1/\beta}.$$

Also, let $W_\beta(D/\sqrt{2\pi}) = \mathbb{E} f_\beta(X^D, D)$ for $D = A, B$. Since for every $\beta > 0$, x and Σ ,

$$\exp \left\{ \max_{1 \leq i \leq N} [x_i - \Sigma_{ii}/2] \right\} \leq f_\beta(x, \Sigma) \leq N^{1/\beta} \exp \left\{ \max_{1 \leq i \leq N} [x_i - \Sigma_{ii}/2] \right\},$$

one has $W_\beta(D/\sqrt{2\pi}) \rightarrow W(D/\sqrt{2\pi})$ as $\beta \rightarrow \infty$. Now, letting $\Sigma(t) = (1-t)\Sigma^A + t\Sigma^B$ and $X(t) = \sqrt{1-t}X^A + \sqrt{t}X^B \sim \mathcal{N}(0, \Sigma(t))$ for $t \in [0, 1]$, define for $u, v \in [0, 1]$:

$$F_\beta(u, v) = \mathbb{E}[f_\beta(X(u), \Sigma(v))],$$

so that $F_\beta(0, 0) = W_\beta(A/\sqrt{2\pi})$ and $F_\beta(1, 1) = W_\beta(B/\sqrt{2\pi})$. It therefore suffices to prove that $t \mapsto F_\beta(t, t)$ is decreasing on $[0, 1]$.

Using Lemma 4.1 below applied to $f = f_\beta(\cdot, \Sigma(t))$,

$$\frac{\partial F_\beta}{\partial u}(t, t) = \frac{1}{2} \sum_{1 \leq i, j \leq N} (\Sigma_{ij}^B - \Sigma_{ij}^A) \mathbb{E} \left[\frac{\partial^2 f_\beta}{\partial x_i \partial x_j}(X(t), \Sigma(t)) \right].$$

An elementary computation shows that, for $1 \leq i, j \leq N$,

$$\begin{aligned} \frac{\partial^2 f_\beta}{\partial x_i \partial x_j}(X(t), \Sigma(t)) &= \left\{ \sum_{i=1}^N e^{\beta[X_i(t) - \Sigma_{ii}(t)/2]} \right\}^{\frac{1}{\beta}-1} \beta e^{\beta[X_i(t) - \Sigma_{ii}(t)/2]} \mathbf{1}(i=j) + \\ &\quad + \left(\frac{1}{\beta} - 1 \right) \left\{ \sum_{i=1}^N e^{\beta[X_i(t) - \Sigma_{ii}(t)/2]} \right\}^{\frac{1}{\beta}-2} \beta e^{\beta[X_i(t) - \Sigma_{ii}(t)/2]} e^{\beta[X_j(t) - \Sigma_{jj}(t)/2]} \\ &= \mathbf{f}(t) \{ \beta \mathbf{p}_i(t) \mathbf{1}(i=j) - (\beta-1) \mathbf{p}_i(t) \mathbf{p}_j(t) \}, \end{aligned}$$

where we let

$$\mathbf{p}_i(t) = \mathbf{p}_i(t; \beta) = \frac{e^{\beta[X_i(t) - \Sigma_{ii}(t)/2]}}{\sum_{k=1}^N e^{\beta[X_k(t) - \Sigma_{kk}(t)/2]}}; \quad \mathbf{f}(t) = \mathbf{f}(t; \beta) = f_\beta(X(t), \Sigma(t)).$$

This leads to

$$\frac{\partial F_\beta}{\partial u}(t, t) = \frac{\beta}{2} \sum_{i=1}^N \mathbb{E}[\mathbf{f}(t) \mathbf{p}_i(t)] (\Sigma_{ii}^B - \Sigma_{ii}^A) - \frac{\beta-1}{2} \sum_{1 \leq i, j \leq N} \mathbb{E}[\mathbf{f}(t) \mathbf{p}_i(t) \mathbf{p}_j(t)] (\Sigma_{ij}^B - \Sigma_{ij}^A). \quad (4.1)$$

On the other hand, by differentiating the expectation defining F_β with respect to its second argument (which is valid since f_β and its differential grow at most exponentially at infinity),

$$\begin{aligned} \frac{\partial F_\beta}{\partial v}(t, t) &= \mathbb{E} \left[\sum_{i=1}^N \frac{1}{\beta} \left\{ \sum_{k=1}^N e^{\beta[X_k(t) - \Sigma_{kk}(t)/2]} \right\}^{\frac{1}{\beta}-1} e^{\beta[X_i(t) - \Sigma_{ii}(t)/2]} (-\beta) \cdot \frac{1}{2} (\Sigma_{ii}^B - \Sigma_{ii}^A) \right] \\ &= -\frac{1}{2} \sum_{i=1}^N \mathbb{E}[\mathbf{f}(t) \mathbf{p}_i(t)] (\Sigma_{ii}^B - \Sigma_{ii}^A). \end{aligned} \quad (4.2)$$

Combine (4.1) and (4.2) to obtain, denoting $\Delta = \Sigma^B - \Sigma^A$,

$$\begin{aligned} \frac{d}{dt} F_\beta(t, t) &= \frac{\partial F_\beta}{\partial u}(t, t) + \frac{\partial F_\beta}{\partial v}(t, t) \\ &= \frac{\beta-1}{2} \left\{ \sum_{i=1}^N \mathbb{E}[\mathbf{f}(t) \mathbf{p}_i(t)] \Delta_{ii} - \sum_{1 \leq i, j \leq N} \mathbb{E}[\mathbf{f}(t) \mathbf{p}_i(t) \mathbf{p}_j(t)] \Delta_{ij} \right\} \\ &= \frac{\beta-1}{4} \sum_{1 \leq i, j \leq N} \mathbb{E}[\mathbf{f}(t) \mathbf{p}_i(t) \mathbf{p}_j(t)] \{ \Delta_{ii} + \Delta_{jj} - 2\Delta_{ij} \} \end{aligned} \quad (4.3)$$

where the last step uses that $\sum_{j=1}^N \mathbf{p}_j(t) = 1$. Now, noting that

$$\Delta_{ii} + \Delta_{jj} - 2\Delta_{ij} = \|\varphi(\theta_i) - \varphi(\theta_j)\|^2 - \|\theta_i - \theta_j\|^2 \leq 0$$

by assumption on φ , we deduce that $\frac{d}{dt} F_\beta(t, t) \leq 0$, which concludes the proof. \square

The proof above used the following standard lemma on Gaussian interpolation (e.g., [Tal11, Lemma 1.3.1 p. 13] or [Ver18, Lemma 7.2.7 p. 162]), proved by Gaussian integration by parts.

Lemma 4.1. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}$ be a twice differentiable function whose Hessian $\nabla^2 f$ has at most exponential growth at infinity. Let Σ^A, Σ^B be positive semi-definite $n \times n$ matrices, and $X^A \sim \mathcal{N}(0, \Sigma^A)$, $X^B \sim \mathcal{N}(0, \Sigma^B)$ be independent. Define $X(t) = \sqrt{1-t}\Sigma^A + \sqrt{t}\Sigma^B$ for $t \in [0, 1]$. Then,*

$$\frac{d}{dt} \mathbb{E}[f(X(t))] = \frac{1}{2} \sum_{1 \leq i, j \leq n} (\Sigma_{ij}^B - \Sigma_{ij}^A) \mathbb{E} \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(X(t)) \right]. \quad (4.4)$$

4.2 Sharp metric characterization of the minimax regret and Wills functional

The comparison principle for the Wills functional and minimax regret (Theorem 4.1) underlines the “metric” nature of these quantities. This qualitative property suggests that these functionals can be understood quantitatively in terms of metric quantities. A fundamental example of this principle comes from the theory of Gaussian processes (see [Tal21] for more information on this topic), where Slepian’s lemma motivated the quantitative study of the Gaussian width of sets in terms of their metric structure [Dud67, Sud76, Fer75, Tal87, Tal21]. This study culminated in the Majorizing Measure (or *generic chaining*) theorem due to Talagrand and Fernique [Tal87] (see [Tal21, Theorem 2.10.1 p. 59] and (4.12)), which provides an explicit metric characterization of the Gaussian width (up to universal constants) through a suitable multi-scale decomposition.

As discussed in Section 3.2, the quantity $R^*(A)$ already has some connection with the Gaussian width, in particular through the inequality $R^*(A) \leq w(A)$. However, this upper bound is far from being sharp in general. In order to obtain more accurate estimates, one must therefore consider additional complexity parameters.

The first such complexity measure is the *local Gaussian width*, defined for any nonempty set $A \subset \mathbf{R}^n$ and $r \geq 0$ by

$$w_A(r) = \sup_{\theta \in A} w(A \cap B(\theta, r)). \quad (4.5)$$

The quantity $w_A(r)$ controls the “local” structure of the set A , at scales smaller than r . Local widths have a natural interpretation in terms of Gaussian processes, since they correspond to the modulus of continuity² of the Gaussian process $(\langle \theta, X \rangle)_{\theta \in A}$ associated to A (where $X \sim \mathcal{N}(0, I_n)$). While this is not immediate from its definition, it follows from the Talagrand-Fernique characterization of the (global) Gaussian width that local widths $w_A(r)$ can be related to the metric structure of A ; see [Tal21, §2.15]. As we will see, the richness of the local structure of A , as measured by local widths, provides a first obstruction to achieving small regret.

The second complexity measure is a “global” one. For any $r \geq 0$, the *covering number* $N(A, r)$ of A at scale r is the number of balls of radius r needed to cover A :

$$N(A, r) = \inf \left\{ N \geq 1 : \exists \theta_1, \dots, \theta_N \in A, A \subset \bigcup_{i=1}^N B(\theta_i, r) \right\}. \quad (4.6)$$

The parameter $\log_2 N(A, r)$ (or more accurately, its upper integral part) has a natural interpretation: it is the number of bits required to describe a finite approximation of A with error at most r ; for this reason, it is sometimes referred to as the metric entropy of A (whose metric

²Specifically, the quantity $w_A(r)$ is connected to a “local” modulus of continuity of the process (with one deterministic endpoint), while the “global” modulus of continuity corresponds to the *a priori* larger quantity $w((A - A) \cap rB_2)$. However, a remarkable property of Gaussian processes discovered by Fernique is that these two moduli of continuity are (in a suitable sense) equivalent up to constants; see [Tal21, p. 82].

character is transparent from its definition). The quantity $\log N(A, r)$ accounts for the global complexity of A , at scales larger than r .

One can associate suitable “fixed points” to both local Gaussian widths and global covering numbers. Specifically, define for any nonempty set $A \subset \mathbf{R}^n$:

$$r_*(A) = \sup \{r \geq 0 : w_A(r) \geq r^2\}; \quad (4.7)$$

$$\tilde{r}(A) = \sup \{r \geq 0 : \log N(A, r) \geq r^2\}. \quad (4.8)$$

(Since $w_A(r) \leq \min(w(A), w(rB_2^n)) \leq \min(w(A), r\sqrt{n})$, one has $r_*(A) \leq \min(w(A), \sqrt{n})$; on the other hand, $\tilde{r}(A)$ is finite if and only if A is bounded, and $\tilde{r}(A) \leq \text{diam}(A)$ since $N(A, r) = 1$ for $r \geq \text{diam}(A)$.)

Both quantities $r_*(A)$ and $\tilde{r}(A)$ are classical complexity measures in high-dimensional statistics. Specifically, fixed points associated to local complexities such as Gaussian widths are known to control the error of the least squares estimator, see [Don90, vdG99, Tsi82, Mas00, Kol11, BBM05, LM13, Cha14] (and references therein) for more information and various extensions. In addition, it is well-known that the fixed point (4.8) associated to covering numbers provides an upper bound on the minimax risk for statistical estimation in various settings [YB99].

The term “fixed points” stems from the fact that, if $g : \mathbf{R}^+ \rightarrow \mathbf{R}^+ \cup \{+\infty\}$ is (say) non-increasing, continuous and neither identically 0 or $+\infty$, then $\sup\{r \geq 0 : g(r) \geq r^2\}$ is the unique fixed point of the function \sqrt{g} . This being said, some care is required in our context as none of the functions w_A and $\log N(A, \cdot)$ (that define the “fixed points” $r_*(A)$ and $\tilde{r}(A)$) satisfies all these properties. The function $\log N(A, \cdot)$ is indeed non-increasing, but not continuous: first, it only takes “quantized” values, and more importantly its value can drop sharply between two values of r of the same order. For instance, one has $\log N(B_2^n, 1) = 0$, but $\log N(B_2^n, 1/2) \asymp n$.

As for the function w_A , it is not decreasing but in fact non-decreasing. However, in the case where $A = K$ is a convex body, one can essentially reduce to the non-decreasing case by considering the function $r \mapsto w_K(r)/r$ which is actually non-increasing (see Lemma 8.2); this ensures in particular that $r_*(K)$ is the unique fixed point of $w_K^{1/2}$. The case where A is a general nonconvex set is more subtle: in this case, w_A needs not be continuous (it can sharply increase at some points), and the function $r \mapsto w_A(r)/r$ may not be decreasing. In this case, there may be several “local solutions” of the fixed-point “equation” (that is, values of r such that for t in a neighborhood of r , one has $w_A(t) \geq t^2$ if $t < r$ and $w_A(t) < t^2$ if $t > r$), of possibly different orders of magnitude. Under such a configuration, by definition $r_*(A)$ corresponds to the supremum of these “local fixed points”.

Some convenient properties of the fixed points (4.7) and (4.8) are gathered in Section 8.1.

With these definitions at hand, we are now in position to state our main quantitative result: a characterization of the minimax regret $R^*(A) = \log W(A/\sqrt{2\pi})$ for general sets A , up to universal constant factors.

Theorem 4.2 (Characterization of the minimax regret). *For every subset $A \subset \mathbf{R}^n$,*

$$\max \left(\frac{r_*(A)^2}{2}, \frac{\tilde{r}(A)^2}{300} \right) \leq R^*(A) \leq 2 \max (r_*(A)^2, \tilde{r}(A)^2). \quad (4.9)$$

Alternatively,

$$\frac{1}{600} \inf_{r>0} \{w_A(r) + \log N(A, r)\} \leq R^*(A) \leq \inf_{r>0} \{w_A(r) + \log N(A, r)\}. \quad (4.10)$$

It follows from Theorem 4.2 that the minimax regret $R^*(A)$ is sharply characterized by a combination of local Gaussian widths and global covering numbers. As noted above, for a given

r , the local width $w_A(r)$ controls the complexity of A at scales smaller than r , while the global covering number $\log N(A, r)$ measures the complexity of A at scales larger than r . In addition, the quantities that appear in Theorem 4.2 can be further expressed explicitly in terms of the metric structure of the set A (see Corollary 4.2 below), therefore making explicit the metric nature of $\mathbf{R}^*(A)$, which is implicit in the comparison inequality of Theorem 4.1.

As a side remark, the equivalence (up to constants) between the quantities in the right-hand sides of (4.9) and (4.10) is not immediate, since the definition of $\max(r_*(A)^2, \tilde{r}(A)^2)$ involves an additional term in r^2 . The “direct” direction is that the infimum in (4.9) is controlled by $\max(r_*(A)^2, \tilde{r}(A)^2)$ (see Section 8.6), while the reverse inequality follows from Theorem 4.2.

Theorem 4.2 can be interpreted as follows. There are at least two basic ways to upper bound $\mathbf{R}^*(A)$. The first one is McMullen’s inequality $\mathbf{R}^*(A) \leq w(A)$, of isoperimetric nature. This inequality is established using tools from Convex Geometry, such as the Alexandrov-Fenchel inequalities between intrinsic volumes as in [McM91] (or the functional Blaschke-Santaló inequality in [AHY21], or the Prékopa-Leindler inequality in Section A.4). It therefore originates from the convex-geometric view of \mathbf{R}^* , in connection with the Wills functional and intrinsic volumes. The second one is the following inequality (see (8.7)): if $A_1, \dots, A_N \subset \mathbf{R}^n$ for some $N \geq 1$, then

$$\mathbf{R}^*\left(\bigcup_{i=1}^N A_i\right) \leq \max_{1 \leq i \leq N} \mathbf{R}^*(A_i) + \log N.$$

This simple but useful inequality is connected to the statistical interpretation of $\mathbf{R}^*(A)$ as a minimax regret; the “mixture”³ argument which establishes it is the basis of the *aggregation* approach to statistical estimation [Bar87, YB99, Cat04, Yan00] and sequential prediction [DMW88, Vov98, LW94]. Combining these two basic inequalities in a natural way (see Section 8.4) leads to the upper bound in (4.10). The matching lower bound in (4.10) from Theorem 4.2 implies that this combination of the “isoperimetric” and “mixture” bounds is optimal in full generality.

Theorem 4.2 is proved in Section 8. In fact, as part of the proof of the lower bound, we also obtain results in the statistical setting (defined in Section 2.1), which we describe in the next section.

Before this, we conclude this section by making the “metric” characterization of $\mathbf{R}^*(A)$ explicit. To this end, we need the following definition (see [Tal21, Definitions 2.7.1 and 2.7.3]):

Definition 4.1 (Admissible sequence, γ_2 functional). Let $A \subset \ell^2$ be a set. An *admissible sequence* of partitions of A is a sequence $(\mathcal{A}_j)_{j \geq 0}$ of partitions of A such that, for each j :

- \mathcal{A}_{j+1} is a refinement of \mathcal{A}_j (i.e., each element of \mathcal{A}_{j+1} is contained in an element of \mathcal{A}_j);
- \mathcal{A}_j contains at most N_j elements, where $N_0 = 1$ and $N_j = 2^{2^j}$ for $j \geq 1$.

Given an admissible sequence, for $\theta \in A$ we denote by $A_j(\theta)$ the unique element of \mathcal{A}_j that contains θ . In addition, for any $p \geq 0$, we define the p -th *truncated γ_2 functional* by

$$\gamma_2^{(p)}(A) = \inf_{(\mathcal{A}_j)_{j \geq 0}} \sup_{\theta \in A} \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta), \quad (4.11)$$

where the infimum is taken over all admissible sequences of partitions of A . (The quantity inside the infimum only depends on $(\mathcal{A}_j)_{j \geq p}$.) The (non-truncated) γ_2 functional is $\gamma_2 = \gamma_2^{(0)}$.

³This terminology comes from the fact that, from the perspective of sequential probability assignment, the predictive distribution that achieves this upper bound is a mixture of the minimax distributions on each set A_i .

The (truncated) γ_2 functionals are explicitly metric, since they are defined in terms of the diameter of suitable multi-scale partitions of A . In particular, it is straightforward to verify that if $\varphi : A \rightarrow \ell^2$ is a contraction, then $\gamma_2^{(p)}(\varphi(A)) \leq \gamma_2^{(p)}(A)$ for every $p \geq 0$.

Talagrand-Fernique’s Majorizing Measure theorem [Tal21, Theorem 2.10.1], a fundamental result in the theory of Gaussian processes, characterizes the magnitude of the Gaussian width in terms of distances through the γ_2 functional. It asserts that there is a universal constant $c > 1$ such that, for any set $A \subset \ell^2$,

$$\frac{1}{c} \gamma_2(A) \leq w(A) \leq c \gamma_2(A). \quad (4.12)$$

The upper bound, due to Fernique, is obtained through an optimal application of the so-called chaining method; the more delicate part of this theorem is the lower bound, due to Talagrand.

Combining Theorem 4.2 with the Majorizing Measure theorem gives an analogous characterization for the regret functional:

Corollary 4.1. *There exists an absolute constant $C > 1$ such that the following holds. For every $A \subset \ell^2$,*

$$\frac{1}{C} \inf_{p \geq 0} \{(2^p - 1) + \gamma_2^{(p)}(A)\} \leq R^*(A) \leq C \inf_{p \geq 0} \{(2^p - 1) + \gamma_2^{(p)}(A)\}. \quad (4.13)$$

Alternatively, let (with the convention that $\sup \emptyset = -\infty$)

$$p^*(A) = \sup \{p \geq 0 : \gamma_2^{(p)}(A) \geq 2^p\}.$$

Define $\gamma^*(A) = \gamma_2(A)$ if $p^*(A) = -\infty$, and $\gamma^*(A) = 2^{p^*(A)}$ otherwise. Then,

$$\frac{1}{C} \gamma^*(A) \leq R^*(A) \leq C \gamma^*(A). \quad (4.14)$$

The proof of Corollary 4.1 is provided in Section 8.7. It is worth noting that, in order to obtain a “monotone” metric characterization like (4.13), it is important that the maximum $\max(\tilde{r}(A), r_*(A))$ is considered, rather than merely $r_*(A)$. Indeed, the quantity $r_*(A)$ alone does *not* necessarily decrease under contractions: if $A = \{0, \sqrt{2\pi}t\}$ with $t \geq 0$, then $r_*(A)^2 = t$ if $t \leq 1/(2\pi)$ but $r_*(A)^2 = 0$ for $t > 1/(2\pi)$.

Starting with Opper and Haussler [OH99], several works obtained upper bounds on the minimax regret for sequential probability assignment through chaining arguments [OH99, CBL01, RS15, BFR20]. While related in spirit to the upper bound in (4.13), these results differ from Corollary 4.1 in that they do not come with matching lower bounds.

While abstract characterizations such as Corollary 4.1 are of significant conceptual interest, in practice the fixed points $r_*(A)$ and $\tilde{r}(A)$ (and thus $R^*(A)$) are often evaluated by more direct means. We refer to Section 7 for some examples.

4.3 Statistical setting: metric characterization of the minimax redundancy

In this section, we consider the “statistical” version of the problem, as defined in Section 2.1. Specifically, for a set $A \subset \mathbf{R}^n$, we defined the *minimax redundancy* $\text{Red}(A) = \text{Red}(\mathcal{P}_A)$ as that of corresponding subset \mathcal{P}_A of the Gaussian model (see (1.4) for the definition of \mathcal{P}_A and (2.2) for that of the minimax redundancy).

Our main result on the minimax redundancy is a characterization of this quantity in terms of covering numbers, which is sharp (up to universal constants) for general subsets of \mathbf{R}^n . Recall the definition (4.8) of the fixed point $\tilde{r}(A)$ associated to global covering numbers.

Theorem 4.3 (Characterization of the minimax redundancy). *For any subset $A \subset \mathbf{R}^n$,*

$$\frac{\tilde{r}(A)^2}{300} \leq \text{Red}(A) \leq 2\tilde{r}(A)^2. \quad (4.15)$$

Alternatively,

$$\frac{1}{600} \inf_{r>0} \{ \log N(A, r) + r^2 \} \leq \text{Red}(A) \leq \inf_{r>0} \{ \log N(A, r) + r^2 \}. \quad (4.16)$$

The proof of Theorem 4.3 (together with that of Theorem 4.2) is provided in Section 8. We note that the upper bound in (4.16) follows from a standard argument, namely aggregation over a net; this is the approach of the classic work of Yang and Barron [YB99]. The contribution in Theorem 4.3 is thus in establishing a matching lower bound.

It is interesting to compare the characterization of the minimax redundancy from Theorem 4.3 to the minimax mean-squared error of Gaussian statistical estimation. Theorem 4.3 states that the redundancy is characterized, for a general set A , by *global* covering numbers. In contrast, it is shown in [Ney22] (see also [Men17] for related results in the random-design setting, and [LC73, IH81, Bir83] for early work in statistics involving local entropy) that the minimax rate of estimation over a convex body K is characterized by *local* covering numbers, namely $N_{\text{loc}}(K, r) = \sup_{\theta \in K} N(K \cap B(\theta, 2r), r)$. The essential difference between the two problems is that the first one is *sequential*, namely involves making successive predictions as observations arrive one by one, while the second problem is non-sequential, in that an estimator of the true mean vector is produced with the knowledge of all observations.

Theorem 4.3 together with prior results on estimation suggest the following principle: optimal rates for sequential estimation problems are governed by global covering numbers, while optimal rates for non-sequential estimation are governed by local covering numbers. A very rough intuitive explanation for this phenomenon is that sequential prediction involves making predictions for different sample sizes (numbers of prior observations), which corresponds to summing individual errors (governed by local covering numbers) at different scales of localization; while on the other hand, global covering numbers behave roughly like a sum of local covering numbers across scales. In fact, neither of these approximations is fully accurate in general, essentially because the maximum of a sum may be smaller than the sum of the maxima; thankfully, this inaccuracy equally affects both terms of the equivalence.

It is also worth comparing the minimax redundancy $\text{Red}(A)$ (corresponding to the average case or the “statistical” setting), scaling as $\tilde{r}(A)^2$, to the minimax regret (corresponding to the worst-case or “deterministic” setting) $\mathbf{R}^*(A)$ scaling as $\max(r_*(A)^2, \tilde{r}(A)^2)$. In cases where $r_*(A) \lesssim \tilde{r}(A)$, the optimal errors are of the same order in both settings; while if $r_*(A) \gg \tilde{r}(A)$ the deterministic variant of the prediction/coding problem is more difficult than the statistical one. The example of ellipsoids (see Section 7.1) shows that both configurations $r_*(A) \ll \tilde{r}(A)$ and $\tilde{r}(A) \ll r_*(A)$ are possible for different sets A .

4.4 Metric estimates for the intrinsic volume sequence

At this point, we have obtained two different characterizations of the minimax regret $\mathbf{R}^*(K)$, when $K \subset \mathbf{R}^n$ is a convex body. The first one (Theorem 2.1) expresses it in terms of intrinsic volumes, while the second one (Theorem 4.2) characterizes it in terms of local Gaussian widths and global covering numbers.

Relating these two characterizations leads to the following result, of purely geometric nature.

Corollary 4.2. *For any convex body $K \subset \mathbf{R}^n$, one has*

$$\frac{1}{600} \inf_{r>0} \{ \log N(K, r) + w_K(r) \} \leq \log \left(\sum_{j=0}^n V_j(K) \right) \leq \sqrt{2\pi} \inf_{r>0} \{ \log N(K, r) + w_K(r) \}. \quad (4.17)$$

Proof. We plug the expression (2.10) of $R^*(K)$ (for the lower bound) and $R^*(\sqrt{2\pi}K)$ (for the upper bound) in terms of intrinsic volumes into the inequalities (4.10). The lower bound follows from the fact that $V_j(K/\sqrt{2\pi}) \leq V_j(K)$, while the upper bound uses that if $r' = \lambda r$ for $\lambda = \sqrt{2\pi} \geq 1$, since $N(\lambda K, r') = N(K, r) \leq \lambda N(K, r)$ and $w_{\lambda K}(r') = \lambda w_K(r)$,

$$\inf_{r' > 0} \{ \log N(\lambda K, r') + w_{\lambda K}(r') \} \leq \lambda \inf_{r > 0} \{ \log N(K, r) + w_K(r) \}. \quad \square$$

Corollary 4.2 relates two types of quantities, both belonging to convex geometry in the broad sense but generally associated to different branches of the topic. Intrinsic volumes are at the core the so-called Brunn-Minkowski theory [Sch13] of classical convex geometry (as well as stochastic geometry [SW08, KR97], through the kinematic formulae), which often involves an exact “isometric” viewpoint. In contrast, quantities such as Gaussian widths and covering numbers play an important role in the modern asymptotic (high-dimensional) theory [AGM15, Pis99], which is often of approximate “isomorphic” character.

Corollary 4.2 gives an isomorphic characterization (that is, a characterization up to constant factors) of the Wills functional of a convex body. That the logarithm is the adequate normalization of the Wills functional to approximate it up to constant factors can be seen from the fact that $W(\frac{\theta}{\sqrt{n}}B_2^n) \rightarrow e^{\sqrt{2\pi}\theta}$ for every $\theta \geq 0$ as $n \rightarrow \infty$ (see (3.6)). Of course, this normalization is also suggested by the statistical interpretation of R^* in connection with universal coding.

The characterization (4.17) provides precise quantitative information on the sequence of intrinsic volumes of a convex body. For instance, by replacing K by a dilation λK with $\lambda \geq 0$, it characterizes up to constants the log-Laplace transform $\lambda \mapsto \log(\sum_{j=0}^n V_j(K)\lambda^j)$ of this sequence. In order to keep this note to a reasonable length, the consequences of these estimates will be developed in a companion paper.

With the background of Corollary 4.2 in mind, a possible interpretation of Corollary 4.1 (characterizing $R^*(A)$ through truncated generic chaining) is that it constitutes a natural variation on the Majorizing Measure characterization of Gaussian width (4.12), as we argue now.

First, a tempting interpretation of the Majorizing Measure theorem is that it characterizes the first intrinsic volume $V_1(K)$ of convex bodies K up to constants. The rationale for this interpretation is that for a compact set $A \subset \mathbf{R}^n$, the convex body $K = \text{conv}(A)$ satisfies $w(A) = w(K) = V_1(K)/\sqrt{2\pi}$, so one may in principle restrict the study of the Gaussian width of general sets to that of convex bodies. Through this lens, a natural question would be to extend this characterization to higher-order intrinsic volumes of convex bodies; that is, to estimate $V_j(K)^{1/j}$ up to constant factors for $j \geq 2$.

However, this interpretation is not fully satisfactory. Indeed, the Majorizing Measure theorem (4.12) relates $w(A)$ to metric properties of the set A itself, rather than of its convex hull $\text{conv}(A)$. In particular, it contains the nontrivial statement that $\gamma_2(A) \asymp \gamma_2(\text{conv}(A))$ for any $A \subset \ell^2$ (which remains to be explained from a purely geometric perspective, see [Tal21, pp. 64–65 and Research Problems 2.11.2–4]); such a statement would be lost by restricting to convex bodies. In addition, convexity does not play any special role in the proof of the Majorizing Measure theorem, or even in the definition of the γ_2 functional which is purely metric. Finally, the quantity $V_j(K)^{1/j}$ is highly sensitive to the exact dimension of K , and can be affected by “mild” operations such as addition of sets: indeed, one may have $V_j(K)^{1/j} = 0$, but $V_j(K \times K) > 0$ (where $K \times K = (K \times \{0\}) + (\{0\} \times K)$). This suggests that a characterization of $V_j(K)^{1/j}$ in terms of objects such as admissible sequences of partitions is unlikely to be at all possible.

In contrast, the Wills functional appears to be a more suitable and regular quantity than individual intrinsic volumes for a metric study of higher-order intrinsic volumes. As noted above, it naturally extends to nonconvex sets (the extension to nonconvex sets is not gratuitous, since $R^*(\text{conv}(A)) \neq R^*(A)$ in general), and as will be seen in Section 5 it interacts well with the

additive structure of the space. From this perspective, instead of individual normalized intrinsic volumes $V_j(K)^{1/j}$ (as a function of j), one considers the regularized quantity $\log(\sum_{j=0}^n V_j(K)\lambda^j)$ as a function of $\lambda \in \mathbf{R}^+$. Corollary 4.1 can therefore be understood as a suitable “generic chaining”-type characterization applying to higher-order intrinsic volumes, expressed in terms of this regularized quantity.

5 Interplay with additive structure and noise dependence

The quantitative estimates from the previous section are of isomorphic nature, in that they hold up to universal constant factors. In this section, we complement these results by studying some exact “isometric” inequalities satisfied by the functional $\mathbf{R}^*(A)$. While versions of those inequalities with additional constant factors could be deduced from the results of Section 4, our aim is to obtain statements with optimal constants, allowing to identify some exact properties such as monotonicity or convexity.

Specifically, this section is dedicated to the study of the interaction between the Wills functional and the affine (additive) structure of ℓ^2 . We will successively consider the behavior of \mathbf{R}^* under convex combinations, sums and differences, and dilations of sets. The last operation has a natural statistical interpretation, which is discussed in Section 5.3.

5.1 Convex combinations

We first consider the behavior of \mathbf{R}^* under convex combinations of sets. The main “concavity” property of the Wills functional was identified by Alonso-Gutiérrez, Hernández Cifre, and Yepes Nicolás [AHY21, Theorem 1.4], who showed that for any convex bodies $K, L \subset \mathbf{R}^n$ and $\lambda \in (0, 1)$,

$$W(\lambda K + (1 - \lambda)L) \geq W(K)^\lambda W(L)^{1-\lambda}. \quad (5.1)$$

The proof of this inequality only relies on the integral expression of the Wills functional (rather than its connection with intrinsic volumes), and does not require convexity. In order to confirm this fact and for the reader’s convenience, we recall the short proof of this inequality due to [AHY21], which relies on the Prékopa-Leindler inequality.

Theorem 5.1. *For every nonempty subsets $A, B \subset \mathbf{R}^n$ and $\lambda \in [0, 1]$, one has*

$$\mathbf{R}^*(\lambda A + (1 - \lambda)B) \geq \lambda \mathbf{R}^*(A) + (1 - \lambda) \mathbf{R}^*(B). \quad (5.2)$$

Proof. Define the functions $f, g, h : \mathbf{R}^n \rightarrow \mathbf{R}^+$ by $f(x) = e^{-\text{dist}^2(x,A)/2}$, $g(y) = e^{-\text{dist}^2(y,B)/2}$ and $h(z) = e^{-\text{dist}^2(z,\lambda A+(1-\lambda)B)/2}$. For any $x, y \in \mathbf{R}^n$, by convexity of the squared norm one has

$$\text{dist}^2(\lambda x + (1 - \lambda)y, \lambda A + (1 - \lambda)B) \leq \lambda \text{dist}^2(x, A) + (1 - \lambda) \text{dist}^2(y, B),$$

namely $h(\lambda x + (1 - \lambda)y) \geq f(x)^\lambda g(y)^{1-\lambda}$. By the Prékopa-Leindler inequality (e.g., [Gar02, Theorem 7.1]), this gives

$$\int_{\mathbf{R}^n} h \geq \left(\int_{\mathbf{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbf{R}^n} g \right)^\lambda.$$

Taking logarithms and recalling the expression (2.11) of \mathbf{R}^* , this establishes (5.2). \square

An application of the concavity property is provided in Appendix A.4. Another one will be presented in the forthcoming note [Mou23].

5.2 Sum and difference bodies

We now turn to the behavior of R^* under Minkowski sums of sets.

The comparison inequality (Theorem 4.1) and the concavity property (Theorem 5.1, due to [AHY21]) are complementary properties of the Wills functional and minimax regret. Combining them gives the following bounds for sum and difference bodies, allowing in particular to restrict to origin-symmetric sets (up to constants).

Proposition 5.1. *For every nonempty subsets $A, B \subset \ell^2$, we have that*

$$R^*(A + B) \leq \sqrt{2}(R^*(A) + R^*(B)). \quad (5.3)$$

In particular, for any $A \subset \ell^2$,

$$R^*(A - A) \leq 2\sqrt{2}R^*(A). \quad (5.4)$$

Proof. Note that $\ell^2 \times \ell^2$ with the natural scalar product is a Hilbert space which identifies with ℓ^2 . In addition, the linear map $\ell^2 \times \ell^2 \rightarrow \ell^2$ given by $(x, y) \mapsto x + y$ is Lipschitz with constant $\sqrt{2}$, since for every $x, y \in \ell^2$ one has

$$\|x + y\| \leq \|x\| + \|y\| \leq \sqrt{2}(\|x\|^2 + \|y\|^2)^{1/2} = \sqrt{2}\|(x, y)\|.$$

Hence, the map $\alpha(x, y) = (x + y)/\sqrt{2}$ is 1-Lipschitz, and by the comparison inequality (Theorem 4.1) it follows that

$$\begin{aligned} R^*(A + B) &= R^*(\alpha(\sqrt{2}A \times \sqrt{2}B)) \leq R^*(\sqrt{2}A \times \sqrt{2}B) = R^*(\sqrt{2}A) + R^*(\sqrt{2}B) \\ &\leq \sqrt{2}(R^*(A) + R^*(B)), \end{aligned}$$

where we used the additivity of R^* over products (Proposition 3.1) and the fact that $R^*(\lambda A) \leq \lambda R^*(A)$ for $\lambda \geq 1$, by concavity of R^* (Theorem 5.1). \square

An interesting aspect of Proposition 5.1 is that it both generalizes (to the case of nonconvex A and $B \neq -A$) and improves (in terms of constants) an inequality deduced from the classical Rogers-Shephard inequality [RS57] on the volume of difference bodies. This inequality [Sch13, Theorem 10.1.4 p. 530] states that for any convex body $K \subset \mathbf{R}^n$,

$$\text{Vol}_n(K - K) \leq \binom{2n}{n} \text{Vol}_n(K) \leq 4^n \text{Vol}_n(K).$$

Plugging this into Kubota's formula (2.6) gives $V_j(K - K) \leq 4^j V_j(K)$ for $0 \leq j \leq n$, so that

$$R^*(K - K) \leq \log \left(\sum_{j=0}^n 4^j V_j(K/\sqrt{2\pi}) \right) = \log \left(\sum_{j=0}^n V_j(4K/\sqrt{2\pi}) \right) = R^*(4K) \leq 4R^*(K)$$

where the last inequality uses concavity of R^* . This is similar (in the special case of convex bodies) to inequality (5.4), but with a worse constant of 4 instead of $2\sqrt{2}$.

However, in the spirit of this section, one may ask whether the $\sqrt{2}$ factor in Proposition 5.1 is necessary. As it turns out, this factor can be removed by a more careful analysis. The corresponding sub-additivity inequality is another ‘‘exact’’ property of the functional R^* , together with the comparison inequality and concavity.

Theorem 5.2. *For any nonempty subsets $A, B \subset \ell^2$, one has*

$$R^*(A + B) \leq R^*(A) + R^*(B). \quad (5.5)$$

In particular, $R^(A - A) \leq 2R^*(A)$.*

The constant 2 in the bound $\mathbf{R}^*(A - A) \leq 2\mathbf{R}^*(A)$ is best possible, since $\mathbf{R}^*(tA) \sim tw(A)$ as $t \rightarrow 0$ (and $w(A - A) = 2w(A)$). Theorem 5.2 can be equivalently stated in terms of the Wills functional, as $W(A + B) \leq W(A)W(B)$ for any $A, B \subset \ell^2$.

Unlike Proposition 5.1, the proof of Theorem 5.2 does not use either the comparison inequality or concavity of \mathbf{R}^* . Instead, it proceeds by showing that $\mathbf{R}^*(A + B) \leq \mathbf{R}^*(A \times B)$, which is achieved by interpolating between the two sets. The proof leverages computations from that of Theorem 4.1, but concludes by exploiting the sum structure rather than distance inequalities.

Remark 5.1 (Link with concavity). A concave function $f : \mathbf{R}^+ \rightarrow \mathbf{R}$ such that $f(0) = 0$ is sub-additive. As such, one may ask whether the same holds for set functionals, allowing to deduce Theorem 5.2 from Theorem 5.1. This is not the case. Theorem 5.2 does imply that the function $f : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ defined by $f(s, t) = \mathbf{R}^*(sA + tB)$ is concave, and clearly $f(0, 0) = 0$; however, these properties do not imply sub-additivity in the bivariate case, as shown by the counter-example $g : (x, y) \mapsto \min(x, y)$ (e.g., $g(1, 0) + g(0, 1) = 0 < g(1, 1) = 1$). For instance, the Brunn-Minkowski inequality [Gar02, Theorem 4.1] implies that the functional $A \mapsto \text{Vol}_n(A)^{1/n}$ (defined over compact sets $A \subset \mathbf{R}^n$) is both concave and *super*-additive; in fact, the two are equivalent for 1-homogeneous functionals. This suggests that concavity and sub-additivity are complementary (rather than similar) additive properties, which may also explain why the proof of Theorem 5.2 has more in common with that of Theorem 4.1 than that of Theorem 5.1.

Proof. By a similar approximation argument as in Theorem 4.1, it suffices to show (5.5) when $A = \{x_i : 1 \leq i \leq N\}$ and $B = \{y_k : 1 \leq k \leq M\}$ are finite subsets of \mathbf{R}^n for some $M, N, n \geq 1$.

We identify $A + B$ with the subset $(A + B) \times \{0\}$ of \mathbf{R}^{2n} , and denote by $A \oplus B = A \times B \subset \mathbf{R}^{2n}$ the orthogonal sum of A, B . By Proposition 3.1, one has $\mathbf{R}^*(A \oplus B) = \mathbf{R}^*(A) + \mathbf{R}^*(B)$, therefore it suffices to show that $\mathbf{R}^*(A + B) \leq \mathbf{R}^*(A \oplus B)$.

We use the interpolation and smoothing arguments and computations from the proof of Theorem 4.1, replacing respectively A, B therein by $A + B$ and $A \oplus B$. Let U, V, W be three independent random vectors with distribution $\mathcal{N}(0, I_n)$. From the Gaussian representation (3.3) and the same arguments as in the proof of Theorem 4.1, defining for $t \in [0, 1]$ and $\beta > 0$:

$$F_\beta(t) = \mathbb{E} \left[\left\{ \sum_{i=1}^N \sum_{k=1}^M \exp \left\{ \beta [X_{ik}(t) - \mathbb{E}X_{ik}(t)^2/2] \right\} \right\}^{1/\beta} \right]$$

$$\text{where } X_{ik}(t) = \sqrt{1-t} \langle x_i + y_k, U \rangle + \sqrt{t} (\langle x_i, V \rangle + \langle y_k, W \rangle),$$

one has $\lim_{\beta \rightarrow \infty} F_\beta(0) = W((A + B)/\sqrt{2\pi})$ and $\lim_{\beta \rightarrow \infty} F_\beta(1) = W((A \oplus B)/\sqrt{2\pi})$. It therefore suffices to show that $F_\beta : [0, 1] \rightarrow \mathbf{R}$ is non-decreasing. In addition, defining respectively

$$\mathbf{p}_{ik}(t) = \mathbf{p}_{ik}(t; \beta) = \frac{e^{\beta[X_{ik}(t) - \mathbb{E}X_{ik}(t)^2/2]}}{\sum_{j=1}^N \sum_{l=1}^M e^{\beta[X_{jl}(t) - \mathbb{E}X_{jl}(t)^2/2]}}; \quad \mathbf{f}(t) = \left\{ \sum_{j=1}^N \sum_{l=1}^M e^{\beta[X_{jl}(t) - \mathbb{E}X_{jl}(t)^2/2]} \right\}^{1/\beta};$$

$$\Delta_{ik,jl} = \mathbb{E}X_{ik}(1)X_{jl}(1) - \mathbb{E}X_{ik}(0)X_{jl}(0),$$

it follows from (4.3) (with appropriate changes in notation) that

$$F'_\beta(t) = \frac{\beta - 1}{4} \sum_{1 \leq i, j \leq N} \sum_{1 \leq k, l \leq M} \mathbb{E}[\mathbf{f}(t)\mathbf{p}_{ik}(t)\mathbf{p}_{jl}(t)] \{ \Delta_{ik,ik} + \Delta_{jl,jl} - 2\Delta_{ik,jl} \}. \quad (5.6)$$

Now, one has successively:

$$\begin{aligned} \mathbb{E}X_{ik}(t)X_{jl}(t) &= (1-t)\langle x_i + y_k, x_j + y_l \rangle + t(\langle x_i, x_j \rangle + \langle y_k, y_l \rangle); \\ \Delta_{ik,jl} &= (\langle x_i, x_j \rangle + \langle y_k, y_l \rangle) - \langle x_i + y_k, x_j + y_l \rangle \\ &= -\langle x_i, y_l \rangle - \langle x_j, y_k \rangle; \end{aligned}$$

$$\begin{aligned}
\Delta_{ik,ik} + \Delta_{jl,jl} - 2\Delta_{ik,jl} &= -2\langle x_i, y_k \rangle - 2\langle x_j, y_l \rangle + 2(\langle x_i, y_l \rangle + \langle x_j, y_k \rangle) \\
&= -2\langle x_i - x_j, y_k - y_l \rangle.
\end{aligned} \tag{5.7}$$

A consequence of (5.7) is that swapping k and l turns the last quantity into its opposite. Plugging (5.7) into the expression (5.6) and using this property gives:

$$\begin{aligned}
F'_\beta(t) &= -\frac{\beta-1}{2} \sum_{1 \leq i, j \leq N} \sum_{1 \leq k, l \leq M} \mathbb{E}[\mathbf{f}(t) \mathbf{p}_{ik}(t) \mathbf{p}_{jl}(t)] \langle x_i - x_j, y_k - y_l \rangle \\
&= -\frac{\beta-1}{4} \cdot \mathbb{E} \left[\mathbf{f}(t) \cdot \sum_{ij,kl} \{ \mathbf{p}_{ik}(t) \mathbf{p}_{jl}(t) - \mathbf{p}_{il}(t) \mathbf{p}_{jk}(t) \} \langle x_i - x_j, y_k - y_l \rangle \right].
\end{aligned} \tag{5.8}$$

Now, since

$$\mathbf{p}_{ik}(t) = \mathbf{f}(t)^{-\beta} \cdot \exp \left\{ \beta [X_{ik}(t) - \mathbb{E}X_{ik}(t)^2/2] \right\},$$

one has

$$\mathbf{p}_{ik}(t) \mathbf{p}_{jl}(t) = \mathbf{f}(t)^{-2\beta} \cdot \exp \left\{ \beta [X_{ik}(t) + X_{jl}(t) - \mathbb{E}X_{ik}(t)^2/2 - \mathbb{E}X_{jl}(t)^2/2] \right\}.$$

In addition,

$$\begin{aligned}
X_{ik}(t) + X_{jl}(t) &= \sqrt{1-t} \langle x_i + x_j + y_k + y_l, U \rangle + \sqrt{t} (\langle x_i + x_j, V \rangle + \langle y_k + y_l, W \rangle) \\
&= X_{il}(t) + X_{jk}(t) \\
\mathbb{E}[X_{ik}(t)^2 + X_{jl}(t)^2] &= (1-t)(\|x_i + y_k\|^2 + \|x_j + y_l\|^2) + t(\|x_i\|^2 + \|x_j\|^2 + \|y_k\|^2 + \|y_l\|^2) \\
&= (\|x_i\|^2 + \|x_j\|^2 + \|y_k\|^2 + \|y_l\|^2) + 2(1-t)(\langle x_i, y_k \rangle + \langle x_j, y_l \rangle).
\end{aligned}$$

As a result, defining

$$\mathbf{f}_{ijkl}(t) = \mathbf{f}(t)^{1-2\beta} \exp \left\{ \beta [X_{ik}(t) + X_{jl}(t) - (\|x_i\|^2 + \|x_j\|^2 + \|y_k\|^2 + \|y_l\|^2)/2] \right\} = \mathbf{f}_{ijlk}(t) \geq 0,$$

one has

$$\mathbf{f}(t) \mathbf{p}_{ik}(t) \mathbf{p}_{jl}(t) = \mathbf{f}_{ijkl}(t) \cdot \exp \left\{ -\beta(1-t)(\langle x_i, y_k \rangle + \langle x_j, y_l \rangle) \right\}.$$

Plugging this expression into (5.8) and using that $\mathbf{f}_{ijkl}(t) = \mathbf{f}_{ijlk}(t)$ yields

$$F'_\beta(t) = -\frac{\beta-1}{4} \sum_{ij,kl} \mathbb{E}[\mathbf{f}_{ijkl}(t)] \left\{ e^{-\beta(1-t)(\langle x_i, y_k \rangle + \langle x_j, y_l \rangle)} - e^{-\beta(1-t)(\langle x_i, y_l \rangle + \langle x_j, y_k \rangle)} \right\} \langle x_i - x_j, y_k - y_l \rangle.$$

Now for $\beta > 0$ and $t \in (0, 1)$, since $\langle x_i - x_j, y_k - y_l \rangle = (\langle x_i, y_k \rangle + \langle x_j, y_l \rangle) - (\langle x_i, y_l \rangle + \langle x_j, y_k \rangle)$, one has $\langle x_i - x_j, y_k - y_l \rangle \geq 0$ if and only if $\langle x_i, y_k \rangle + \langle x_j, y_l \rangle \geq \langle x_i, y_l \rangle + \langle x_j, y_k \rangle$, if and only if $e^{-\beta(1-t)(\langle x_i, y_k \rangle + \langle x_j, y_l \rangle)} \leq e^{-\beta(1-t)(\langle x_i, y_l \rangle + \langle x_j, y_k \rangle)}$. Therefore

$$\left\{ e^{-\beta(1-t)(\langle x_i, y_k \rangle + \langle x_j, y_l \rangle)} - e^{-\beta(1-t)(\langle x_i, y_l \rangle + \langle x_j, y_k \rangle)} \right\} \langle x_i - x_j, y_k - y_l \rangle \leq 0$$

and thus $F'_\beta(t) \geq 0$, which concludes the proof. \square

5.3 Dilations and dependence on noise and sample size

We conclude this section by considering dilations of sets. This operation has a natural statistical interpretation, in terms of varying the “noise level” of “sample size”, as described below.

Previously, we have considered the class of Gaussians with unit variance, namely distributions $\mathcal{N}(\theta, I_n)$ with $\theta \in \mathbf{R}^n$. One may also consider Gaussians with general variance $\sigma^2 > 0$. Specifically, for $\theta \in \mathbf{R}^n$, let $p_{\theta, \sigma^2}(x) = (2\pi\sigma^2)^{-n/2} \exp(-\|x - \theta\|^2/(2\sigma^2))$ be the density of the

distribution $\mathcal{N}(\theta, \sigma^2 I_n)$. Likewise, for a subset $A \subset \mathbf{R}^n$, denote $\mathcal{P}_{A, \sigma^2} = \{p_{\theta, \sigma^2} : \theta \in A\}$ the associated subset of the Gaussian model with variance σ^2 , and define

$$\mathbf{R}^*(A, \sigma^2) = \mathbf{R}^*(\mathcal{P}_{A, \sigma^2}) \quad (5.9)$$

the associated minimax regret (with respect to $\mathcal{P}_{A, \sigma^2}$) for sequential probability assignment. Now, since rescaling by σ^{-1} changes the model $\mathcal{P}_{A, \sigma^2}$ into $\mathcal{P}_{A/\sigma, 1} = \mathcal{P}_{A/\sigma}$, we have that

$$\mathbf{R}^*(A, \sigma^2) = \mathbf{R}^*(A/\sigma). \quad (5.10)$$

Hence, dilations of A can be understood as varying the *noise level* σ^2 .

Another interpretation comes from the “repeated” variant of the coding problem. Here, instead of predicting or coding a single vector $x = (x_1, \dots, x_d) \in \mathbf{R}^d$, one aims to predict a sequence of vectors $x^{(1)}, \dots, x^{(n)} \in \mathbf{R}^d$ for some “sample size” $n \geq 1$. In addition, given a set $A \subset \mathbf{R}^d$ (which induces the model \mathcal{P}_A on \mathbf{R}^d for a single vector), the model on sequences of vectors is obtained by modeling the n vectors as i.i.d. random vectors distributed as p_θ for some $\theta \in A$. This amounts to sequential probability assignment over $(\mathbf{R}^d)^n \simeq \mathbf{R}^{nd}$, with model

$$\{p_\theta^{\otimes n} : \theta \in A\} = \mathcal{P}_{A_n}$$

where $A_n = \{(\theta, \dots, \theta) : \theta \in A\} \subset \mathbf{R}^{nd}$. Denote $\mathbf{R}_n^*(A) = \mathbf{R}^*(A_n)$ the associated minimax regret. Note that $A_n = \pi_n(\sqrt{n}A)$ where $\pi_n : \mathbf{R}^d \rightarrow \mathbf{R}^{nd}$ is the linear isometry defined by $\pi_n(\theta) = (\theta/\sqrt{n}, \dots, \theta/\sqrt{n})$, hence by Proposition 3.1 one has

$$\mathbf{R}_n^*(A) = \mathbf{R}^*(A_n) = \mathbf{R}^*(\sqrt{n}A). \quad (5.11)$$

In particular, $\mathbf{R}_n^*(A) = \mathbf{R}^*(A, \sigma^2)$ with $\sigma = 1/\sqrt{n}$.

The quantities $\mathbf{R}^*(A, \sigma^2)$ and $\mathbf{R}_n^*(A)$ are normalized by total code-length (log-likelihood), but alternative normalizations are also meaningful. Since $\ell(p_{\theta, \sigma^2}, x) = \|\theta - x\|^2 / (2\sigma^2)$ for $\theta, x \in \mathbf{R}^n$, multiplying log-loss by σ^2 makes it proportional to the standard squared error for densities in $\mathcal{P}_{A, \sigma^2}$, suggesting the alternative normalization $\sigma^2 \cdot \mathbf{R}^*(A, \sigma^2)$. Likewise, $\mathbf{R}_n^*(A)$ measures regret for the sum of code-lengths over the vectors $x^{(1)}, \dots, x^{(n)}$, another natural quantity being the average (per-vector) regret $\mathbf{R}_n^*(A)/n$ —equal to $\sigma^2 \mathbf{R}^*(A, \sigma^2)$ for $\sigma = 1/\sqrt{n}$.

It is natural to expect that the total regret $\mathbf{R}_n^*(A)$ increases with n , namely that $\mathbf{R}^*(A, \sigma^2)$ decreases with σ (as σ^2 increases, distributions p_{θ, σ^2} with $\theta \in A$ become “closer” to each other in an information-theoretic sense, so the model $\mathcal{P}_{A, \sigma^2}$ becomes smaller). Conversely, one may expect that the average regret $\mathbf{R}_n^*(A)/n$ decreases with n , namely that $\sigma^2 \cdot \mathbf{R}^*(A, \sigma^2)$ increases with σ . The following proposition confirms that both of these monotonicity properties hold, and shows that the second one can in fact be strengthened.

Proposition 5.2. *For any subset $A \subset \ell^2$, the function $t \mapsto \mathbf{R}^*(tA)$ is increasing on \mathbf{R}_+^* , while the function $t \mapsto \mathbf{R}^*(tA)/t$ is decreasing.*

In particular, $\mathbf{R}_n^(A)$ increases with n (resp. $\mathbf{R}^*(A, \sigma^2)$ decreases with σ), while $\mathbf{R}_n^*(A)/\sqrt{n}$ decreases with n (resp. $\sigma \cdot \mathbf{R}^*(A, \sigma^2)$ increases with σ).*

Proof. In the case where $A = K$ is a convex body, the fact that $\mathbf{R}^*(tK)$ increases with t follows directly from either the expression in terms of intrinsic volumes, or from monotonicity with respect to inclusion (Proposition 3.1) as one can assume that $0 \in K$ up to translating. However, both of these arguments are specific to the convex case.

In the general case, the fact that $\mathbf{R}^*(tA)$ increases with t follows from the comparison inequality (Theorem 4.1), since for $0 < s \leq t$ the contraction $x \mapsto (s/t)x$ maps tA onto sA .

We now turn to the second claim. By concavity of \mathbf{R}^* (Theorem 5.1) applied to dilations of A , the map $t \mapsto \mathbf{R}^*(tA)$ is concave on \mathbf{R}^+ , and equals 0 at 0. As a result, the slope $\mathbf{R}^*(tA)/t$ decreases with t . Alternatively, by the sub-additivity property (Theorem 5.2), the map $t \mapsto \mathbf{R}^*(tA)$ is sub-additive, and since it is also increasing it is continuous, so $\mathbf{R}^*(tA)/t$ decreases. \square

6 Comparison with classical asymptotics of coding

In this section, we relate and contrast our results to classical asymptotic results on universal coding in information theory.

Classical asymptotics. A seminal result of Rissanen [Ris96] provides a precise asymptotic expansion of the minimax regret in the fixed-dimensional regime. It implies that for a fixed compact and sufficiently smooth subset $A \subset \mathbf{R}^d$ with non-empty interior, as $n \rightarrow \infty$,

$$\mathbf{R}_n^*(A) = \frac{d}{2} \log \left(\frac{n}{2\pi} \right) + \log \text{Vol}_d(A) + o(1). \quad (6.1)$$

In fact, the asymptotic expansion (6.1) extends to more general families of distributions smoothly parameterized by a compact subset of \mathbf{R}^d or manifold, where the “volume” in (6.1) is now defined with respect to the volume form associated to the Fisher information metric (see [Ris96] and [Grü07, Chapter 7]). This general result—which is based on a quadratic approximation of the log-likelihood, i.e. a Laplace approximation of Shtarkov’s integral (2.4)—is an instance of the “local asymptotic normality” phenomenon in asymptotic parametric statistics [LCY00]. Within the Gaussian setting, Proposition 3.4 (together with the identity $\mathbf{R}_n^*(A) = \mathbf{R}^*(\sqrt{n}A)$) shows that the expansion (6.1) holds for any compact set $A \subset \mathbf{R}^d$ with $\text{Vol}_d(A) > 0$, without additional regularity assumptions.

Rissanen’s expansion had a deep influence on the theory of universal coding, data compression and model selection. It plays a key role within the “Minimum Description Length” (MDL) paradigm to coding and estimation, proposed by Rissanen and expounded in [Ris85, BRY98, Grü07]. In particular, the quantity (6.1), sometimes called “stochastic complexity”, has been suggested as a notion of model complexity and used as a complexity penalty in model selection.

An interesting feature of the expansion (6.1) is that it features the volume of the parameter set $A \subset \mathbf{R}^d$, albeit as a constant second-order term dominated by the $\frac{d}{2} \log n$ term. Hence, in the asymptotic regime, the minimax regret $\mathbf{R}_n^*(A) \sim \frac{d}{2} \log n$ is essentially determined by the dimension, with a lower-order term depending on the volume.

A limitation of this result is that it is purely asymptotic, in that it only holds for fixed A (and thus d) as $n \rightarrow \infty$. This rules out the modern “high-dimensional” regime, where the dimension may be comparable to, or larger than, the sample size n ; as such, it does not describe what happens for a general sample size n . In addition, it does not specify how large n should be for the expansion (6.1) to be accurate.

Comparison with estimates for convex bodies. When $A = K \subset \mathbf{R}^d$ is a convex body and for a general sample size n , Proposition 2.1 implies that, whenever $n \geq 4/w(K)^2$ (arguably the statistically meaningful regime, since $w(K) \gtrsim 1$ when the model is “non-trivial”),

$$\mathbf{R}_n^*(K) \asymp \max_{1 \leq j \leq d} \log V_j \left(\sqrt{\frac{n}{2\pi}} K \right) = \max_{1 \leq j \leq d} \left\{ \frac{j}{2} \log \left(\frac{n}{2\pi} \right) + \log V_j(K) \right\}. \quad (6.2)$$

This expression is reminiscent of (6.1), but with the *volume* replaced by the maximum *intrinsic volume*. The estimate (6.2) is also non-asymptotic, namely it holds for a general sample size n .

This result allows to address the two aforementioned limitations of the asymptotic expansion. First, it allows to quantify for which sample sizes n the asymptotic expansion is accurate: it suffices that the d^{th} intrinsic volume in the maximum of (6.2) dominates, which amounts (by the unimodality properties of the intrinsic volume sequence, see Section A.1) to $V_d(\sqrt{n/2\pi}K) \geq$

$V_{d-1}(\sqrt{n/2\pi}K)$. By homogeneity of intrinsic volumes and the link with volume and surface, assuming that $\text{Vol}_d(K) > 0$ this is equivalent to

$$n \geq 8\pi \cdot \left(\frac{\text{surface}_{d-1}(\partial K)}{\text{Vol}_d(K)} \right)^2. \quad (6.3)$$

Second, the estimate (6.2) shows that, for a general sample size n that does not satisfy the ‘‘large sample’’ condition (6.3), ‘‘stochastic complexity’’ is characterized by a lower-dimensional intrinsic volume rather than the volume.

7 Examples

In this section, we leverage the general results of Section 4 to derive explicit characterizations of the minimax redundancy and regret in some natural examples.

7.1 Ellipsoids

We now consider the case of general ellipsoids in \mathbf{R}^n . Ellipsoids arise naturally in the regression context described at the beginning of Section 2.3, when the underlying class of functions \mathcal{F} is the ball of a reproducing kernel Hilbert space [Aro50, SC08]. By invariance under affine isometries, one may restrict to non-degenerate axis-aligned ellipsoids, of the following form. For $a = (a_1, \dots, a_n)$ with $a_1 \geq \dots \geq a_n > 0$, define the ellipsoid $E_a \subset \mathbf{R}^n$ by

$$E_a = \left\{ (\theta_1, \dots, \theta_n) \in \mathbf{R}^n : \sum_{i=1}^n \frac{\theta_i^2}{a_i^2} \leq 1 \right\}. \quad (7.1)$$

First, recall the standard Kahane-Khintchine estimate on the Gaussian width of ellipsoids; it also follows (with different constants) from the concentration property of the Gaussian measure.

Lemma 7.1. *For any $a \in \mathbf{R}_+^n$, one has*

$$\frac{1}{\sqrt{3}} \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \leq w(E_a) \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2}. \quad (7.2)$$

Proof. Let $X = (X_1, \dots, X_n) \sim \mathcal{N}(0, I_n)$. Letting $Z = \left(\sum_{i=1}^n a_i^2 X_i^2 \right)^{1/2}$ and $\|Z\|_p = \mathbb{E}[Z^p]^{1/p}$,

$$w(E_a) = \mathbb{E} \sup_{\theta \in E_a} \langle \theta, X \rangle = \mathbb{E} \left[\left(\sum_{i=1}^n a_i^2 X_i^2 \right)^{1/2} \right] = \|Z\|_1 \leq \|Z\|_2 = \left(\sum_{i=1}^n a_i^2 \right)^{1/2}.$$

On the other hand, since $\mathbb{E}X_i^2 X_j^2 \leq 3$ for $1 \leq i, j \leq n$ (with equality if $i = j$), we have

$$\|Z\|_4 = \mathbb{E} \left[\left(\sum_{i=1}^n a_i^2 X_i^2 \right)^2 \right]^{1/4} \leq \left[3 \sum_{1 \leq i, j \leq n} a_i^2 a_j^2 \right]^{1/4} = 3^{1/4} \left(\sum_{i=1}^n a_i^2 \right)^{1/2},$$

namely $\|Z\|_4 \leq 3^{1/4} \|Z\|_2$. Finally, Hölder’s inequality implies that $\|Z\|_2 \leq \|Z\|_1^{1/3} \|Z\|_4^{2/3}$, so that $\|Z\|_1 / \|Z\|_2 \geq (\|Z\|_2 / \|Z\|_4)^2 \geq 1/\sqrt{3}$. \square

In the case of $E_a = B_2^n$ (namely $a_1 = \dots = a_n = 1$), the proof above with the exact value $\mathbb{E}Z^4 = n^2 + 2n$ gives

$$\left(1 + \frac{2}{\sqrt{n}} \right)^{-1/2} \sqrt{n} \leq w(B_2^n) \leq \sqrt{n}. \quad (7.3)$$

The following estimate for local widths of ellipsoids is standard, see e.g. [Men03] for related results on local Rademacher averages of ellipsoids.

Lemma 7.2. For every $r > 0$,

$$\frac{1}{\sqrt{3}} \left(\sum_{i=1}^n \min(a_i^2, r^2) \right)^{1/2} \leq w_{E_a}(r) \leq \sqrt{2} \left(\sum_{i=1}^n \min(a_i^2, r^2) \right)^{1/2}. \quad (7.4)$$

Proof. Since E_a is convex and symmetric, by Lemma 8.2 one has $w_{E_a}(r) = w(E_a \cap rB_2)$. Let $L \subset \mathbf{R}^n$ denote the centered axis-aligned ellipsoid with half-lengths $\min(a_i, r)$, $1 \leq i \leq n$. From the inequalities, for every $\theta \in \mathbf{R}^n$,

$$\max \left(\sum_{i=1}^n \frac{\theta_i^2}{a_i^2}, \sum_{i=1}^n \frac{\theta_i^2}{r^2} \right) \leq \sum_{i=1}^n \frac{\theta_i^2}{\min(a_i^2, r^2)} \leq \sum_{i=1}^n \left(\frac{\theta_i^2}{a_i^2} + \frac{\theta_i^2}{r^2} \right) \leq 2 \max \left(\sum_{i=1}^n \frac{\theta_i^2}{a_i^2}, \sum_{i=1}^n \frac{\theta_i^2}{r^2} \right),$$

one deduces the inclusions $L \subset E_a \cap rB_2 \subset \sqrt{2}L$, so that $w(L) \leq w_{E_a}(r) \leq \sqrt{2}w(L)$. Applying Lemma 7.1 concludes the proof. \square

We will also need the following covering number estimates for ellipsoids. These estimates are classical; see e.g. [CS90, Theorem 3.4.2 p. 119] for an essentially equivalent result, stated in terms of entropy numbers of operators between Hilbert spaces.

Lemma 7.3. For any $r > 0$,

$$\log N(E_a, 5r) \leq \sum_{i: a_i \geq 2r} \log \left(\frac{a_i}{r} \right) \leq \log N(E_a, r). \quad (7.5)$$

Proof. Recall that $a_1 \geq \dots \geq a_n$. Let $d = \max\{i : a_i \geq 2r\}$ (with the convention $d = 0$ if $\max_i a_i < 2r$), and let $\Pi : \mathbf{R}^n \rightarrow \mathbf{R}^d$ the orthogonal projection on the first d coordinates.

For the second inequality, note that if $\theta_1, \dots, \theta_N$ form an r -cover of E_a , then $\Pi\theta_1, \dots, \Pi\theta_N$ form an r -cover of $E' = \Pi(E_a)$. The claimed inequality is then obtained by noting that

$$\kappa_d \prod_{i=1}^d a_i = \text{Vol}_d(E') \leq \sum_{j=1}^N \text{Vol}_d(B_d(\Pi\theta_j, r)) = N\kappa_d r^d.$$

For the first inequality, denote $E' = \Pi(E_a)$, and identify \mathbf{R}^d with $\mathbf{R}^d \times \{0\} \subset \mathbf{R}^n$. Note that if $\theta \in E_a$, then $\sum_{i>d} \theta_i^2/r^2 \leq \sum_{i>d} \theta_i^2/a_i^2 \leq 1$ namely $\|\theta - \Pi(\theta)\| \leq r$. Thus $E_a \subset E' + rB_2^n$, which implies that $N(E_a, (K+1)r) \leq N(E', Kr)$ for any $K > 1$. Recalling that N_p refers to packing numbers, inequality (8.5) provides $N(E', Kr) \leq N_p(E', Kr)$. Let $N = N_p(E', Kr)$ and $\theta_1, \dots, \theta_N$ form a Kr -packing of $E' \subset \mathbf{R}^d$. Then, the balls $B(\theta_j, Kr/2)$, $1 \leq j \leq N$, are pairwise disjoint and contained in $E' + Kr/2B_2^d \subset (1 + K/4)E'$, so that

$$\text{Vol}_d((1 + K/4)E') = \kappa_d \prod_{i=1}^d ((1 + K/4)a_i) \geq \sum_{j=1}^N \text{Vol}_d(B(\theta_j, r/2)) = N\kappa_d (Kr/2)^d.$$

This implies, for $K = 4$,

$$\log N(E_a, 5r) \leq \log N_p(E', 4r) \leq \sum_{i=1}^d \log \left(\frac{(1 + K/4)a_i}{(K/2)r} \right) = \sum_{i: a_i \geq 2r} \log \left(\frac{a_i}{r} \right). \quad \square$$

With these estimates at hand, together with the consequence (8.1) of Lemma 8.1, we can characterize the two fixed points associated to E_a :

$$\begin{aligned} \frac{1}{6} \inf_{r>0} \left\{ \sum_{i=1}^n \min \left(\frac{a_i^2}{r^2}, 1 \right) + r^2 \right\} &\leq r_*(E_a)^2 \leq 2 \inf_{r>0} \left\{ \sum_{i=1}^n \min \left(\frac{a_i^2}{r^2}, 1 \right) + r^2 \right\} \\ \frac{1}{2} \inf_{r>0} \left\{ \sum_{i: a_i \geq 2r} \log \left(\frac{a_i}{r} \right) + r^2 \right\} &\leq \tilde{r}(E_a)^2 \leq 5 \inf_{r>0} \left\{ \sum_{i: a_i \geq 2r} \log \left(\frac{a_i}{r} \right) + r^2 \right\}. \end{aligned}$$

(For the second fixed point, we used the equivalent representation in terms of $w(E_a \cap rB_2)^2/r^2 = w(r^{-1}E_a \cap B_2)^2$ described in Section A.3.) The minimax regret for ellipsoids is then characterized as follows:

Proposition 7.1. *One has:*

$$\frac{1}{6000} \inf_{r>0} \left\{ \sum_{i=1}^n \log \left(1 + \frac{a_i^2}{r^2} \right) + r^2 \right\} \leq \mathbf{R}^*(E_a) \leq 10 \inf_{r>0} \left\{ \sum_{i=1}^n \log \left(1 + \frac{a_i^2}{r^2} \right) + r^2 \right\}. \quad (7.6)$$

Furthermore, there exists $\lambda \geq 0$ such that the density q_λ of the Gaussian distribution

$$\mathcal{N} \left(0, \sum_{i=1}^n \left(1 + \frac{a_i^2}{\lambda} \right) e_i e_i^\top \right)$$

where (e_1, \dots, e_n) is the canonical basis of \mathbf{R}^n , satisfies, for any sequence $y = (y_1, \dots, y_n) \in \mathbf{R}^n$,

$$\ell(q_\lambda, y) - \inf_{\theta \in E_a} \ell(p_\theta, y) \leq 3000 \mathbf{R}^*(E_a). \quad (7.7)$$

Proposition 7.1 characterizes (up to constants) the magnitude of the minimax regret, and thus the Wills functional, on ellipsoids. The second part of this statement asserts that a suitable Gaussian predictive distribution achieves minimax-optimal regret over any ellipsoid. Gaussian distributions are particularly convenient in practice from the perspective of sequential prediction (described in the introduction) due to the fact that conditioning a Gaussian measure on a linear projection reduces to simple matrix computations. Since our focus is on the functional \mathbf{R}^* rather than practical implementation of prediction schemes, we do not expand further on this point.

Proof of Proposition 7.1. From the characterization (4.9) of Theorem 4.2 as well as the previous fixed points estimates for ellipsoids, after re-arranging one has

$$\begin{aligned} \frac{1}{3600} \inf_{r>0} \left\{ \sum_{i=1}^n \min \left(\frac{a_i^2}{r^2}, 1 \right) + \sum_{i: a_i \geq 2r} 2 \log \left(\frac{a_i}{r} \right) + r^2 \right\} &\leq \mathbf{R}^*(E_a) \\ &\leq 5 \inf_{r>0} \left\{ \sum_{i=1}^n \min \left(\frac{a_i^2}{r^2}, 1 \right) + \sum_{i: a_i \geq 2r} 2 \log \left(\frac{a_i}{r} \right) + r^2 \right\}. \end{aligned} \quad (7.8)$$

Now, for $a_i < 2r$,

$$(\log 5)^{-1} \log \left(1 + \frac{a_i^2}{r^2} \right) \leq \min \left(\frac{a_i^2}{r^2}, 1 \right) \leq \frac{a_i^2}{r^2} \leq (\log 2)^{-1} \log \left(1 + \frac{a_i^2}{r^2} \right),$$

while for $a_i \geq 2r$ (so that $\min(a_i^2/r^2, 1) = 1$)

$$\log \left(1 + \frac{a_i^2}{r^2} \right) \leq 2 \log \left(\frac{a_i}{r} \right) + 1 \leq 2 \log \left(1 + \frac{a_i^2}{r^2} \right).$$

Summing these two inequalities gives

$$(\log 5)^{-1} \sum_{i=1}^n \log \left(1 + \frac{a_i^2}{r^2} \right) \leq \sum_{i=1}^n \min \left(\frac{a_i^2}{r^2}, 1 \right) + \sum_{i: a_i \geq 2r} 2 \log \left(\frac{a_i}{r} \right) \leq 2 \log \left(1 + \frac{a_i^2}{r^2} \right).$$

Plugging this into (7.8) (and using that $3600 \log 5 \leq 6000$) gives (7.6).

For the second statement, we use the following identity shown (with different but equivalent “kernel” notation) by [SKF08], which is also implicitly used in [Vov01]: for every $y \in \mathbf{R}^n$,

$$\ell(q_\lambda, y) - \inf_{\theta \in \mathbf{R}^n} \left\{ \ell(p_\theta, y) + \frac{\lambda}{2} \sum_{i=1}^n \frac{\theta_i^2}{a_i^2} \right\} = \frac{1}{2} \sum_{i=1}^n \log \left(1 + \frac{a_i^2}{\lambda} \right). \quad (7.9)$$

This expression for the regret can be obtained through the identity

$$\inf_{\theta \in \mathbf{R}^n} \left\{ \ell(p_\theta, y) + \frac{\lambda}{2} \sum_{i=1}^n \frac{\theta_i^2}{a_i^2} \right\} = \inf_{\theta \in \mathbf{R}^n} \left\{ \frac{1}{2} \|\theta - y\|^2 + \frac{\lambda}{2} \sum_{i=1}^n \frac{\theta_i^2}{a_i^2} \right\} = \frac{1}{2} \sum_{i=1}^n \frac{y_i^2}{1 + a_i^2/\lambda},$$

which corresponds to $\ell(q_\lambda, y)$ up to the constant in the r.h.s. of (7.9). It follows from (7.9) and the definition of E_a that, for every $y \in \mathbf{R}^n$,

$$\ell(q_\lambda, y) - \inf_{\theta \in E_a} \ell(p_\theta, y) \leq \frac{1}{2} \sum_{i=1}^n \log \left(1 + \frac{a_i^2}{\lambda} \right) + \frac{\lambda}{2}.$$

This upper bound together with the minimax regret lower bound in (7.6) imply (7.7). \square

A similar characterization and optimality result hold for the minimax redundancy:

Proposition 7.2. *The minimax redundancy on the ellipsoid E_a satisfies:*

$$\frac{1}{600} \inf_{r>0} \left\{ \sum_{i: a_i \geq 2r} \log \left(\frac{a_i}{r} \right) + r^2 \right\} \leq \text{Red}(E_a) \leq 5 \inf_{r>0} \left\{ \sum_{i: a_i \geq 2r} \log \left(\frac{a_i}{r} \right) + r^2 \right\}. \quad (7.10)$$

In addition, there exists $\lambda \geq 0$ such that the density \tilde{q}_λ of the Gaussian distribution

$$\mathcal{N} \left(0, \sum_{i=1}^n \left(1 + \frac{a_i^2}{\lambda} \mathbf{1}(a_i \geq 2\sqrt{\lambda}) \right) e_i e_i^\top \right)$$

satisfies

$$\sup_{\theta \in E_a} \text{KL}(p_\theta, \tilde{q}_\lambda) \leq 1200 \text{Red}(E_a). \quad (7.11)$$

Proof. The upper and lower bounds in (7.10) are obtained by combining Theorem 4.3 with the covering number upper bound of Lemma 7.3 and the lower bound on $\tilde{r}(E_a)^2$ above.

To bound the maximal redundancy of \tilde{q}_λ , we use the following identity: if q denotes the density of the distribution $\mathcal{N}(0, \Sigma)$ (for a positive semi-definite $n \times n$ matrix Σ) then

$$\text{KL}(p_\theta, q) = \frac{1}{2} \left(\log \det \Sigma - n + \text{Tr}(\Sigma^{-1}) \right) + \frac{1}{2} \langle \Sigma^{-1} \theta, \theta \rangle.$$

In particular, for every $\theta \in E_a$, using that $1 + (a_i^2/\lambda) \mathbf{1}(a_i \geq 2\sqrt{\lambda}) \geq a_i^2/(4\lambda)$ for all i ,

$$\begin{aligned} \text{KL}(p_\theta, \tilde{q}_\lambda) &= \frac{1}{2} \sum_{i: a_i \geq 2\sqrt{\lambda}} \left\{ \log \left(1 + \frac{a_i^2}{\lambda} \right) - 1 + \frac{\lambda}{a_i^2} \right\} + \frac{1}{2} \sum_{i=1}^n \frac{\theta_i^2}{1 + (a_i^2/\lambda) \mathbf{1}(a_i \geq 2\sqrt{\lambda})} \\ &\leq \frac{1}{2} \sum_{i: a_i \geq 2\sqrt{\lambda}} \log \left(1 + \frac{a_i^2}{\lambda} \right) + 2\lambda \sum_{i=1}^n \frac{\theta_i^2}{a_i^2} \\ &\leq \frac{1}{2} \sum_{i: a_i \geq 2r} \log \left(1 + \frac{a_i^2}{r^2} \right) + 2r^2 \end{aligned}$$

where we let $r = \sqrt{\lambda}$. Inequality (7.11) is then a consequence of this upper bound and of the lower bound in (7.10). \square

7.2 The ℓ^1 ball

Another “canonical” example is given by the ℓ^1 ball. This class plays an important role in modern high-dimensional statistics and signal processing [FR13, BvdG11, Tsy09, Kol11, Wai19].

Here, we consider the set $\alpha B_1^d = \{\theta \in \mathbf{R}^d : \sum_{i=1}^d |\theta_i| \leq \alpha\}$ for some $\alpha > 0$ and $d \geq 2$. The minimax error for Gaussian statistical estimation over this set was characterized in [DJ94], while here we consider the hardness of universal coding.

In order to characterize the fixed point $\tilde{r}(\alpha B_1^d)$, we first recall the notion of *entropy numbers*, which are closely related to covering numbers. Specifically, for any integer $k \geq 0$ and set $A \subset \mathbf{R}^n$, the k -th entropy number $e_k(A)$ is defined as

$$e_k(A) = \inf \left\{ r > 0 : \exists x_1, \dots, x_{2^k} \in \mathbf{R}^n, A \subset \bigcup_{i=1}^{2^k} B(x_i, r) \right\} = \inf \{ r > 0 : N(A, r) \leq 2^k \}.$$

Using the connection with covering numbers in the last expression, one readily verifies that

$$\tilde{r}(A)^2 \asymp \inf_{k \geq 0} \{k + e_k(A)^2\}.$$

Now, it was shown in [Sch84] that

$$e_k(B_1^d) \asymp \begin{cases} 1 & \text{if } k \leq \log d \\ \sqrt{\log(ed/k)/k} & \text{if } \log d \leq k \leq d \\ 2^{-k/d} d^{-1/2} & \text{if } k \geq d. \end{cases}$$

From these estimates (together with Theorem 4.3) and by distinguishing cases, one deduces that

$$\text{Red}(\alpha B_1^d) \asymp \tilde{r}(\alpha B_1^d)^2 \asymp \begin{cases} \alpha^2 & \text{if } \alpha \leq \sqrt{\log d} \\ \alpha \sqrt{\log(ed/\alpha)} & \text{if } \sqrt{\log d} \leq \alpha \leq d \\ d \log(e\alpha/d) & \text{if } \alpha \geq d. \end{cases} \quad (7.12)$$

In addition, it is known from [GLM⁺07] that

$$w(\alpha B_1^d \cap r B_2^d) \asymp \begin{cases} \alpha \sqrt{\log(ed \min(r^2/\alpha^2, 1))} & \text{if } r \geq \alpha/\sqrt{d} \\ r\sqrt{d} & \text{if } r \leq \alpha/\sqrt{d}. \end{cases}$$

This implies (see e.g. [LM13, p. 19]) that

$$r_*(\alpha B_1^d)^2 \asymp \begin{cases} \alpha \sqrt{\log(ed \min(1/\alpha, 1))} & \text{if } \alpha \leq d \\ d & \text{if } \alpha \geq d. \end{cases} \quad (7.13)$$

The fixed points estimates (7.12) and (7.13), together with Theorem 4.2, allow to conclude that

$$\mathbf{R}^*(\alpha B_1^d) \asymp \begin{cases} \alpha \sqrt{\log d} & \text{if } \alpha \leq 1 \\ \alpha \sqrt{\log(ed/\alpha)} & \text{if } 1 \leq \alpha \leq d \\ d \log(e\alpha/d) & \text{if } \alpha \geq d. \end{cases} \quad (7.14)$$

An upper bound on the minimax regret essentially corresponding to the second regime in (7.14) (up to a non-explicit dependence on the norm inside the logarithm, possibly due to the asymptotic regime implicitly considered there) was obtained in [MY19]. This upper bound matches (in the regime $1 \leq \alpha \leq d^{1-\varepsilon}$) the order of the minimax estimation error from [DJ94].

8 Proof of Theorems 4.2 and 4.3

8.1 Fixed points and local widths

For convenience, we gather here some basic properties of the “fixed points” we encounter.

Lemma 8.1. *For any function $g : \mathbf{R}^+ \rightarrow \mathbf{R}^+ \cup \{+\infty\}$ which is not identically $+\infty$, define*

$$\phi(g) = \sup \{r \geq 0 : g(r) \geq r^2\} \in [0, +\infty].$$

1. *Let $g, h : \mathbf{R}^+ \rightarrow \mathbf{R}^+ \cup \{+\infty\}$. Then:*

- (a) *if $g \leq h$, then $\phi(g) \leq \phi(h)$;*
- (b) *$\phi(\max(g, h)) = \max\{\phi(g), \phi(h)\}$.*

2. *Assume now that $r \mapsto g(r)/r$ is non-increasing on \mathbf{R}_+^* . Then,*

- (a) *$\phi(g) < \infty$, and $r^2 < g(r)$ for $r < \phi(g)$ while $r^2 > g(r)$ for $r > \phi(g)$;*
- (b) *for any $\lambda \geq 1$, one has $\phi(\lambda g) \leq \lambda \phi(g)$;*
- (c) *for any $\lambda \geq 1$, let $g_\lambda : r \mapsto g(r/\lambda)$; then, one has $\phi(g_\lambda) \leq \lambda \phi(g)$.*

3. *Finally, assume that g is non-increasing. Then,*

$$\phi(g)^2 \leq \inf_{r \geq 0} \{g(r) + r^2\} \leq 2\phi(g)^2.$$

In particular, if g, h are such that $g(r)/r$ and $h(r)/r$ are non-increasing in r , and if there are constants $C_1, C_2 \geq 1$ and $c_1, c_2 \leq 1$ such that, for every $r \geq 0$,

$$c_1 h(r/c_2) \leq g(r) \leq C_1 h(r/C_2),$$

then

$$c_1 c_2 \phi(h) \leq \phi(g) \leq C_1 C_2 \phi(h). \quad (8.1)$$

Proof. Properties 1(a-b) follow directly from the definition of ϕ . Property 2(a) comes from the fact that $\phi(g) = \sup\{r > 0 : r \leq g(r)/r\}$ (with the convention that $\sup \emptyset = 0$), and that $r \mapsto g(r)/r$ is non-increasing (and thus bounded on $[x, +\infty)$ where $g(x) < \infty$), while r is increasing and unbounded. To prove property 2(b), let $r < \phi(\lambda g)$ be arbitrary. By 2(a) and since $g(r)/r \leq g(r/\lambda)/(r/\lambda)$,

$$r^2 < \lambda g(r) \leq \lambda^2 g\left(\frac{r}{\lambda}\right)$$

namely $(r/\lambda)^2 < g(r/\lambda)$ and thus $r/\lambda \leq \phi(g)$. Letting $r \rightarrow \phi(g)$ we get $\phi(\lambda g) \leq \lambda \phi(g)$. Likewise, to prove 2(c), let $r < \phi(g_\lambda)$ be arbitrary. By 2(a),

$$\left(\frac{r}{\lambda}\right)^2 \leq r^2 < g\left(\frac{r}{\lambda}\right)$$

so that $r/\lambda \leq \phi(g)$ which concludes the proof. Finally, let us prove property 3. Recall that $\phi(g) < \infty$ by 2(a). To prove the second inequality, let $r > \phi(g)$ be arbitrary, so that $g(r) < r$ and thus $g(r) + r^2 \leq 2r^2$, and let $r \rightarrow \phi(g)$. We now prove the first inequality, namely $\phi(r) + r^2 \geq \phi(g)^2$ for any $r \geq 0$. If $r \geq \phi(g)$, then $\phi(r) + r^2 \geq r^2 \geq \phi(g)^2$. If $r < \phi(g)$, let $r' \in (r, \phi(g))$ be arbitrary. Since ϕ is non-increasing and $r < r' < \phi(r)$, one has $\phi(r) + r^2 \geq \phi(r) \geq \phi(r') \geq (r')^2$; letting $r' \rightarrow \phi(g)$ concludes. \square

In the convex case, local Gaussian widths have the following convenient properties:

Lemma 8.2. *Let $K \subset \mathbf{R}^n$ be a convex body.*

1. *The function $r \mapsto w_K(r)/r$ is decreasing;*
2. *If K is symmetric, then $w_K(r) = w(K \cap rB_2)$.*

Proof. We start with the first point. Up to translating K , assume that $0 \in K$. Let $0 < r \leq r'$ and $\lambda = r'/r \geq 1$. For every $\theta \in K$,

$$\begin{aligned} w(K \cap B(\theta, r')) &= w((K - \theta) \cap \lambda r B_2) \\ &= \lambda w(\lambda^{-1}(K - \theta) \cap r B_2) \\ &\leq \lambda w_{\lambda^{-1}K}(r) \\ &\leq \lambda w_K(r), \end{aligned}$$

where the last inequality comes from the fact that $\lambda^{-1}K \subset K$ (as K is convex and contains 0). Taking the supremum over $\theta \in K$ and substituting for λ leads to $w_K(r')/r' \leq w_K(r)/r$.

For the second point, fix $r \geq 0$ and let $g(\theta) = w(K \cap B(\theta, r))$ for $\theta \in K$, so that $w_K(r) = \sup_{\theta \in K} g(\theta)$. Since K is symmetric, the function g is even, namely $g(-\theta) = g(\theta)$. It is also concave since for $\theta, \theta' \in K$ and $\lambda \in [0, 1]$ one has

$$K \cap B(\lambda\theta + (1 - \lambda)\theta', r) \supset \lambda[K \cap B(\theta, r)] + (1 - \lambda)[K \cap B(\theta', r)],$$

so that (using that $w(A + B) = w(A) + w(B)$, by definition of the Gaussian width)

$$g(\lambda\theta + (1 - \lambda)\theta') \geq \lambda g(\theta) + (1 - \lambda)g(\theta').$$

This implies, for every $\theta \in K$, that $g(\theta) = (g(\theta) + g(-\theta))/2 \leq g((\theta - \theta)/2) = g(0)$. \square

8.2 Upper bound on the redundancy

We now include the proof of the upper bounds of Theorem 4.3 on the minimax redundancy. The upper bound is classical and due to [YB99], who used it to further bound the rate of estimation.

Fix $A \subset \mathbf{R}^n$. If $N(A, r) = \infty$ for any $r > 0$ (i.e., A is unbounded), then the right-hand sides of both (4.15) and (4.16) are infinite and the upper bounds hold. Let $r > 0$ be arbitrary such that $N = N(A, r)$ is finite. Let $\theta_1, \dots, \theta_N \in A$ be an r -cover of A , and define the density

$$q_r = \frac{1}{N} \sum_{i=1}^N p_{\theta_i}.$$

For every $\theta \in A$ and $1 \leq i \leq N$, using that $q_r \geq p_{\theta_i}/N$ one has

$$\text{KL}(p_\theta, q_r) = \int_{\mathbf{R}^n} p_\theta \log \left(\frac{p_\theta}{q_r} \right) \leq \int_{\mathbf{R}^n} p_\theta \log \left(\frac{N \cdot p_\theta}{p_{\theta_i}} \right) = \text{KL}(p_\theta, p_{\theta_i}) + \log N.$$

Using the identity $\text{KL}(p_\theta, p_\vartheta) = \|\theta - \vartheta\|^2/2$ for $\theta, \vartheta \in \mathbf{R}^n$, taking the infimum over i , and using that $\{\theta_1, \dots, \theta_N\}$ form an r -cover of A , this gives:

$$\text{KL}(p_\theta, q_r) \leq \frac{1}{2} \min_{1 \leq i \leq N} \|\theta - \theta_i\|^2 + \log N \leq \frac{r^2}{2} + \log N(A, r).$$

Since $r > 0$ was arbitrary, this gives

$$\text{Red}(A) \leq \inf_{r>0} \left\{ \log N(A, r) + \frac{r^2}{2} \right\} \leq 2\tilde{r}(A)^2, \quad (8.2)$$

where the second inequality uses point 3 of Lemma 8.1, applied to $g(r) = \log N(A, r)$.

8.3 Lower bound on the redundancy

The proof starts by lower-bounding the minimax redundancy by a mutual information.

Lemma 8.3. *For any $N \geq 1$ and $\theta_1, \dots, \theta_N \in A$, denoting $\bar{p}_N = N^{-1} \sum_{i=1}^N p_{\theta_i}$, one has:*

$$\text{Red}(A) \geq \frac{1}{N} \sum_{i=1}^N \text{KL}(p_{\theta_i}, \bar{p}_N).$$

Proof. For any density q on \mathbf{R}^n , one has

$$\sup_{\theta \in A} \text{KL}(p_{\theta}, q) \geq \frac{1}{N} \sum_{i=1}^N \text{KL}(p_{\theta_i}, q) = \frac{1}{N} \sum_{i=1}^N \text{KL}(p_{\theta_i}, \bar{p}_N) + \text{KL}(\bar{p}_N, q), \quad (8.3)$$

where the last identity can be verified directly. Lower-bounding $\text{KL}(\bar{p}_N, q) \geq 0$ for any q and taking the infimum of the l.h.s. over q concludes the proof of Lemma 8.3. \square

The following lemma is an important ingredient of the proof. It states that if A contains a sufficiently large and separated subset, then its minimax redundancy is large.

Lemma 8.4. *Assume that $\theta_1, \dots, \theta_N \in \mathbf{R}^n$ are such that $\|\theta_i - \theta_j\| \geq r$ for $i \neq j$, with $r^2 \geq 32 \log N$. Then,*

$$\frac{1}{N} \sum_{i=1}^N \text{KL}(p_{\theta_i}, \bar{p}_N) \geq \log N - 3. \quad (8.4)$$

Proof. In order to prove (8.4), it suffices to prove that $\text{KL}(p_{\theta_i}, \bar{p}_N) \geq \log N - 3$ for each $i = 1, \dots, N$. By symmetry, it suffices to prove it for $i = 1$; and up to translating A , one may assume that $\theta_1 = 0$. In particular, $\|\theta_i\| \geq r$ for $2 \leq i \leq N$. We then have, letting $X \sim \mathcal{N}(0, I_n)$,

$$\begin{aligned} \text{KL}(p_0, \bar{p}_N) &= \int_{\mathbf{R}^n} p_0 \log \left(\frac{p_0}{N^{-1} \sum_{i=1}^N p_{\theta_i}} \right) = \log N - \int_{\mathbf{R}^n} p_0 \log \left(\sum_{i=1}^N \frac{p_{\theta_i}}{p_0} \right) \\ &= \log N - \mathbb{E} \left[\log \left(1 + \sum_{i=2}^N e^{\langle \theta_i, X \rangle - \|X_i\|^2/2} \right) \right]. \end{aligned}$$

Hence, $\text{KL}(p_0, \bar{p}_N) = \log N - \mathbb{E}[Z]$, where

$$Z = \log \left(1 + \sum_{i=2}^N e^{\langle \theta_i, X \rangle - \|X_i\|^2/2} \right).$$

It therefore suffices to show that $\mathbb{E}[Z] \leq 3$. We write, using that $\|\theta_i\| \geq r$ for $2 \leq i \leq N$,

$$\begin{aligned} Z &\leq \log \left\{ 1 + N \max_{2 \leq i \leq N} \exp \left[\langle \theta_i, X \rangle - \frac{\|\theta_i\|^2}{2} \right] \right\} \\ &= \log \left\{ 1 + \exp \left[\left(\log N - \frac{r^2}{32} \right) + \max_{2 \leq i \leq N} \left(\langle \theta_i, X \rangle - \frac{15\|\theta_i\|^2}{32} \right) \right] \right\} \\ &\leq \log \left\{ 1 + \exp \left[\max_{2 \leq i \leq N} \left(\langle \theta_i, X \rangle - \frac{15\|\theta_i\|^2}{32} \right) \right] \right\} \end{aligned}$$

where the last inequality is due to the assumption that $r^2 \geq 32 \log N$. Now, for $2 \leq i \leq N$, one has $\langle \theta_i, X \rangle \sim \mathcal{N}(0, \|\theta_i\|^2)$, so by a Gaussian tail bound (e.g., [BLM13, p. 22]), for any $\delta \in (0, 1)$:

$$\mathbb{P}(\langle \theta_i, X \rangle \geq \|\theta_i\| \sqrt{2 \log(1/\delta)}) \leq \delta.$$

By a union bound, if $\delta \in (0, 1)$, with probability at least $1 - \delta$, for $i = 2, \dots, N$,

$$\langle \theta_i, X \rangle < \|\theta_i\| \sqrt{2 \log(N/\delta)} \leq \|\theta_i\| \sqrt{2 \log N} + \|\theta_i\| \sqrt{2 \log(1/\delta)} \leq \frac{\|\theta_i\|^2}{4} + \|\theta_i\| \sqrt{2 \log(1/\delta)},$$

where the last inequality uses that $\log N \leq r^2/32 \leq \|\theta_i\|^2/32$. Hence, with probability $1 - \delta$,

$$\begin{aligned} \max_{2 \leq i \leq N} \left(\langle \theta_i, X \rangle - \frac{15\|\theta_i\|^2}{32} \right) &\leq \max_{2 \leq i \leq N} \left(\|\theta_i\| \sqrt{2 \log(1/\delta)} + \frac{\|\theta_i\|^2}{4} - \frac{15\|\theta_i\|^2}{32} \right) \\ &\leq \sup_{\rho \geq 0} \left(\rho \sqrt{2 \log(1/\delta)} - \frac{7\rho^2}{32} \right) = \frac{16}{7} \log(1/\delta). \end{aligned}$$

Putting the previous inequalities together, with probability at least $1 - \delta$,

$$Z \leq \log(1 + \delta^{-16/7}) \leq \log(2\delta^{-16/7}) = \log 2 + \frac{16}{7} \log(1/\delta).$$

Integrating the tails gives:

$$\mathbb{E}[Z] \leq \int_0^1 \left(\log 2 + \frac{16}{7} \log(1/\delta) \right) d\delta = \log 2 + \frac{16}{7} \leq 3.$$

This concludes the proof. \square

We note that Lemma 8.4 could also be proved using a lower bound on the mutual information in terms of a log-Laplace transform of the Rényi divergence from Haussler and Opper [HO97, Theorem 2]; this lower bound implies in particular a lower bound on the minimax redundancy in terms of Hellinger covering numbers [PW22, Theorem 32.5 p. 530]. The Hellinger distance is particularly convenient by virtue of being a distance (allowing, for instance, to relate packing and covering numbers); on the other hand, in the Gaussian case relaxing the Rényi divergence into a Hellinger distance leads to suboptimal lower bounds in general. We thank Yanjun Han for bringing Haussler and Opper's inequality to our attention.

With these lemmas at hand, we conclude the proof of the lower bound in Theorem 4.3. Let $\tilde{r} = \tilde{r}(A) \geq 0$ (which may be infinite, in case $A \subset \mathbf{R}^n$ is unbounded).

First case: $\tilde{r} \gtrsim 1$. We first assume that \tilde{r} is large enough, namely $\tilde{r} \geq 12$.

For any $r > 0$, let $N_p(A, r)$ denote the packing number of A at scale r , defined as the supremum of all $N \geq 1$ for which there exist $\theta_1, \dots, \theta_N \in A$ such that $\|\theta_i - \theta_j\| > r$ for $1 \leq i, j \leq N$, $i \neq j$. Packing and covering numbers are related through the following standard inequalities (e.g., [Ver18, Lemma 4.2.8 p. 83]):

$$N_p(A, 2r) \leq N(A, r) \leq N_p(A, r). \quad (8.5)$$

Define

$$\tilde{r}_p = \sup \left\{ r \geq 0 : \log N_p(A, r) \geq r^2 \right\}.$$

By inequality (8.5) and the consequence (8.1) of Lemma 8.1, we have that

$$\tilde{r}_p/2 \leq \tilde{r} \leq \tilde{r}_p.$$

This implies that $\tilde{r}_p > 12$. Now, let $12 < r < \tilde{r}_p$ be arbitrary, and let $N \geq 1$ be the largest integer such that $\log N \leq r^2/32$. Since $r \geq 12$, one has $N \geq 90$ and $r^2/32 \leq \log(N+1) \leq \frac{\log 91}{\log 90} \log N$ so that $\log N \geq r^2/33$.

In addition, since $r < \tilde{r}_p$ and $\log N_p(A, r)$ is non-increasing, one has $\log N_p(A, r) \geq r^2 \geq r^2/32 \geq \log N$. Hence, $N \leq N_p(A, r)$ and one may find $\theta_1, \dots, \theta_N \in A$ such that $\|\theta_i - \theta_j\| \geq r$ for $i \neq j$. By Lemmas 8.3 and 8.4, it follows that

$$\text{Red}(A) \geq \log N - 3 \geq \frac{r^2}{33} - 3 \geq \frac{r^2}{110}$$

since $r \geq 12$. Letting $r \rightarrow \tilde{r}_p$, we obtain $\text{Red}(A) \geq \tilde{r}_p^2/110 \geq \tilde{r}^2/110$.

Second case: $\tilde{r} \lesssim 1$. We now handle the complementary case $0 \leq \tilde{r} < 12$. (The derivation below is only needed to lower-bound the redundancy; for the minimax regret $R^*(A)$ and Wills functional, it is implied by the stronger lower bound of Section 8.5.)

First, note that since $\log N(A, r)$ is non-increasing, if $\tilde{r} = 0$ then for any $r > 0$ one has $\log N(A, r) = 0$ namely $N(A, r) = 1$. This implies that A has a single element $\theta \in \mathbf{R}^n$ and thus (taking $q = p_\rho$ in the expression (2.2) of $\text{Red}(A)$) $\text{Red}(A) = 0$.

Now, assume that $\tilde{r} > 0$. For $0 < r < \tilde{r}$, one has $\log N(A, r) \geq r^2 > 0$ so $N(A, r) \geq 2$. It follows that A contains two elements θ_1, θ_2 such that $\rho = \|\theta_1 - \theta_2\| > r$. By Lemma 8.3,

$$\text{Red}(A) \geq \frac{1}{2} \left\{ \text{KL} \left(p_{\theta_1}, \frac{p_{\theta_1} + p_{\theta_2}}{2} \right) + \text{KL} \left(p_{\theta_2}, \frac{p_{\theta_1} + p_{\theta_2}}{2} \right) \right\} = \text{KL} \left(p_0, \frac{p_0 + p_\rho}{2} \right),$$

where the reduction to $(\theta_1, \theta_2) = (0, \rho) \in \mathbf{R} \times \mathbf{R}$ uses the invariance and independence properties of Gaussian measures. Let $\|p - q\|_{\text{TV}} = \frac{1}{2} \int |p - q|$ denote the total variation distance between two densities p, q . By Pinsker's inequality (e.g., [Tsy09, Lemma 2.5 p. 88]),

$$\text{KL} \left(p_0, \frac{p_0 + p_\rho}{2} \right) \geq 2 \cdot \left\| p_0 - \frac{p_0 + p_\rho}{2} \right\|_{\text{TV}}^2 = \frac{1}{2} \cdot \|p_0 - p_\rho\|_{\text{TV}}^2.$$

Now, a direct computation gives, denoting $p(x) = p_0(x) = e^{-x^2/2}/\sqrt{2\pi}$,

$$\|p_0 - p_\rho\|_{\text{TV}} = \frac{1}{2} \left(\int_{-\infty}^{\rho/2} (p(x) - p(x - \rho)) dx + \int_{\rho/2}^{\infty} (p(x - \rho) - p(x)) dx \right) = 2 \int_0^{\rho/2} p(x) dx$$

Since p is positive and decreasing on \mathbf{R}^+ , $\tau(\rho) = \|p_0 - p_\rho\|_{\text{TV}}$ is increasing and $\tau(\rho)/\rho$ decreasing with respect to ρ . Recalling that $\rho \geq r$ and $r < \tilde{r} \leq 12$, this gives

$$\|p_0 - p_\rho\|_{\text{TV}} \geq \tau(r) \geq \frac{\tau(12)}{12} \cdot r \geq \frac{r}{12.1}.$$

Combining the previous inequalities and recalling that $0 < r < \tilde{r}$ was arbitrary, we deduce that

$$\text{Red}(A) \geq \sup_{0 < r < \tilde{r}} \frac{1}{2} \left(\frac{r}{12.1} \right)^2 \geq \frac{\tilde{r}^2}{300}.$$

We have thus proved that, in all cases, $\text{Red}(A) \geq \tilde{r}(A)^2/300$. The alternative lower bound (4.16) comes from the fact that, by Lemma 8.1,

$$\tilde{r}(A)^2 \geq \frac{1}{2} \inf_{r > 0} \{ \log N(A, r) + r^2 \}.$$

8.4 First upper bound on the regret

We now turn to the study of the minimax regret $R^*(A)$, starting from the upper bound

$$R^*(A) \leq \inf_{r>0} \left\{ w_A(r) + \log N(A, r) \right\}. \quad (8.6)$$

This upper bound is obtained by naturally combining two distinct basic bounds, namely McMullen's "isoperimetric" bound (3.5) and another "mixture" argument already used in the upper bound on the redundancy in Section 8.2.

We start with the mixture bound: if $A \subset \bigcup_{i=1}^N A_i$ for some $N \geq 1$ and $A_1, \dots, A_N \subset \mathbf{R}^n$,

$$\begin{aligned} R^*(A) &\leq \log \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-\min_{1 \leq i \leq N} \text{dist}^2(x, A_i)/2} dx \right) \\ &\leq \log \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} \sum_{i=1}^N e^{-\text{dist}^2(x, A_i)/2} dx \right) = \log \left(\sum_{i=1}^N \exp(R^*(A_i)) \right) \end{aligned}$$

so that

$$R^*(A) \leq \max_{1 \leq i \leq N} R^*(A_i) + \log N. \quad (8.7)$$

Now, for any $r > 0$, let $N = N(A, r)$ and $\{\theta_1, \dots, \theta_N\} \subset A$ an r -cover of A . One may write

$$A = \bigcup_{i=1}^N [A \cap B(\theta_i, r)].$$

Applying the mixture bound (8.7) followed by McMullen's upper bound (Proposition 3.2) gives

$$R^*(A) \leq \max_{1 \leq i \leq N} R^*(A \cap B(\theta_i, r)) + \log N \leq \sup_{\theta \in A} w(A \cap B(\theta, r)) + \log N(A, r),$$

which is precisely (8.6), since $r > 0$ is arbitrary.

8.5 Lower bound on the regret

We now prove the following lower bound: for any $A \subset \mathbf{R}^n$

$$R^*(A) \geq \max \left(\frac{r_*(A)^2}{2}, \frac{\tilde{r}(A)^2}{300} \right). \quad (8.8)$$

First, the lower bound in terms of $\tilde{r}(A)$ follows from the lower bound for the redundancy proved in Section 8.3: by (2.5) and (4.15),

$$R^*(A) \geq \text{Red}(A) \geq \frac{\tilde{r}(A)^2}{300}.$$

It remains to show the lower bound in terms of $r_*(A)$. Let $r > 0$ and $\theta \in A$ be arbitrary. By monotonicity of $R^*(A)$ under inclusion (Proposition 3.1), one has:

$$R^*(A) \geq R^*(A \cap B(\theta, r)).$$

On the other, since $A \cap B(\theta, r) \subset B(\theta, r)$, the lower bound (3.4) implies that

$$R^*(A \cap B(\theta, r)) \geq w(A \cap B(\theta, r)) - \frac{r^2}{2}.$$

Combining the previous inequalities and taking the supremum over $\theta \in A$ yields, for every $r > 0$,

$$\mathbf{R}^*(A) \geq \sup_{\theta \in A} w(A \cap B(\theta, r)) - \frac{r^2}{2} = w_A(r) - \frac{r^2}{2}. \quad (8.9)$$

Now, let $r < r_*(A)$ be arbitrary. By definition (4.7) of $r_*(A)$, there exists $r' \geq r$ such that $w_A(r') \geq r'^2$. By (8.9) applied to r' , one has

$$\mathbf{R}^*(A) \geq w_A(r') - \frac{r'^2}{2} \geq \frac{r'^2}{2} \geq \frac{r^2}{2}.$$

Taking the supremum over $r < r_*(A)$ gives $\mathbf{R}^*(A) \geq r_*(A)^2/2$, concluding the proof of (8.8).

8.6 Relating the upper and lower bounds

At this point, we have shown that

$$\max\left(\frac{r_*(A)^2}{2}, \frac{\tilde{r}(A)^2}{300}\right) \leq \mathbf{R}^*(A) \leq \inf_{r>0} \left\{w_A(r) + \log N(A, r)\right\}.$$

To show that both upper and lower bounds above are sharp and conclude the proof of Theorem 4.2, it remains to bound (up to constants) the rightmost term by the leftmost term.

In order to lighten notation, we keep dependence on A implicit, and let $r_* = r_*(A)$, $\tilde{r} = \tilde{r}(A)$, and for $r > 0$ denote $N(r) = N(A, r)$ and $w(r) = w_A(r)$. It remains to show that $\inf_{r>0} \{w(r) + \log N(r)\} \lesssim \max(r_*^2, \tilde{r}^2)$.

If $\max(r_*, \tilde{r}) = +\infty$, the inequality clearly holds, so we assume that $\max(r_*, \tilde{r}) < \infty$. Let $r > \max(r_*, \tilde{r})$ be arbitrary. By definition of r_* and \tilde{r} , one has

$$w(r) < r^2 \quad \text{and} \quad \log N(r) < r^2,$$

so that $w(r) + \log N(r) < 2r^2$. Taking the infimum over $r > \max(r_*, \tilde{r})$ leads to

$$\inf_{r>0} \left\{w_A(r) + \log N(A, r)\right\} \leq 2 \max(r_*(A)^2, \tilde{r}(A)^2). \quad (8.10)$$

Combining inequalities (8.6), (8.8) and (8.10) concludes the proof of Theorem 4.2.

8.7 Generic chaining characterization

We now establish the alternative characterizations of Corollary 4.1 in terms of admissible sequences and truncated γ_2 functionals.

We start with the proof the upper bound in (4.13). Let $\mathcal{A} = (\mathcal{A}_j)_{j \geq 0}$ be an admissible sequence of partitions of A , and let $p \in \mathbf{N}$ be arbitrary. By the upper bound (8.7) applied to the partition \mathcal{A}_p of A (with at most N_p elements) combined with the Gaussian width upper bound of Proposition 3.2, one has

$$\mathbf{R}^*(A) \leq \log N_p + \max_{A' \in \mathcal{A}_p} \mathbf{R}^*(A') \leq \log N_p + \max_{A' \in \mathcal{A}_p} w(A').$$

Note that $\log N_p = 0$ if $p = 0$ and $\log N_p = 2^p \log 2$ if $p \geq 1$, so that $\log N_p \leq 2(2^p - 1)$ for all $p \geq 0$. In addition, for every $A' \in \mathcal{A}_p$, the sequence $(\mathcal{A}_j^{A'})_{j \geq 0}$ of partitions of A' defined by $\mathcal{A}_j^{A'} = \{A'\}$ for $j \leq p - 1$, and $\mathcal{A}_j^{A'} = \{B \in \mathcal{A}_j : B \subset A'\}$ for $j \geq p$, is an admissible sequence.

Also, for $\theta \in A' \subset A$, the element of $\mathcal{A}_j^{A'}$ containing θ is $A' = A_p(\theta)$ if $j \leq p-1$ and $A_j(\theta)$ if $j \geq p$. By the upper bound in (4.12), one has

$$\begin{aligned} w(A') &\leq c \sup_{\theta \in A'} \left(\sum_{j=0}^{p-1} 2^{j/2} \text{diam } A_p(\theta) + \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta) \right) \\ &= c \sup_{\theta \in A'} \left(\frac{2^{p/2} - 1}{\sqrt{2} - 1} \text{diam } A_p(\theta) + \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta) \right) \\ &\leq \frac{2c}{\sqrt{2} - 1} \sup_{\theta \in A'} \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta). \end{aligned}$$

Putting these bounds together gives

$$\begin{aligned} R^*(A) &\leq 2(2^p - 1) + \frac{2c}{\sqrt{2} - 1} \max_{A' \in \mathcal{A}_p} \sup_{\theta \in A'} \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta) \\ &\leq 2(\sqrt{2} + 1)c \left\{ (2^p - 1) + \sup_{\theta \in A} \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta) \right\}. \end{aligned}$$

Since \mathcal{A} and $p \in \mathbf{N}$ were arbitrary, taking the infimum over both gives the upper bound in (4.13).

We now turn to the lower bound in (4.13). It follows from the lower bound in Theorem 4.2 together with the lower bound in the Majorizing Measure theorem (see (4.12)).

By the lower bound (4.10) of Theorem 4.2, it suffices to show the following: there exists an absolute constant C' such that, for every $r > 0$, there exists $p \in \mathbf{N}$ and an admissible sequence $(\mathcal{A}_j)_{j \geq 0}$ such that

$$(2^p - 1) + \sup_{\theta \in A} \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta) \leq C' \left\{ \log N(A, r) + w_A(r) \right\}. \quad (8.11)$$

We now fix $r > 0$.

If $N(A, r) = 1$, we let $p = 0$ and then (8.11) reduces to the lower bound (4.12) in the Majorizing Measure theorem.

Now, assume that $N = N(A, r) \geq 2$ and let $p \geq 1$ be the smallest integer such that $N_p \geq N$, so that (by definition of N_p) $\log N \leq \log N_p \leq 2 \log N$. Let $\{\theta_1, \dots, \theta_N\} \subset A$ be an r -cover of A , so that $A = \bigcup_{i=1}^N A_i$ where $A_i = A \cap B(\theta_i, r)$. Note that $w(A_i) \leq w_A(r)$ by definition of w_A .

For every $i = 1, \dots, N$, by the lower bound (4.12) in the Majorizing Measure theorem, there exists an admissible sequence $(\mathcal{A}_{ij})_{j \geq 0}$ of partitions of A_i such that

$$\sup_{\theta \in A_i} \sum_{j \geq p} 2^{j/2} \text{diam } A_{ij}(\theta) \leq \sup_{\theta \in A_i} \sum_{j \geq 0} 2^{j/2} \text{diam } A_{ij}(\theta) \leq c w(A_i) \leq c w_A(r).$$

Now, define the following admissible sequence $\mathcal{A} = (\mathcal{A}_j)_{j \geq 0}$ of partitions of A : for $j \geq p$, let $\mathcal{A}_j = \bigcup_{i=1}^N \mathcal{A}_{i,j-p}$, and define \mathcal{A}_j arbitrarily for $0 \leq j \leq p-1$, since the left-hand side of (8.11) only depends on $(\mathcal{A}_j)_{j \geq p}$. This is indeed an admissible sequence: note that $\mathcal{A}_p = \{A_1, \dots, A_N\}$, so $|\mathcal{A}_p| = N \leq N_p$; while for $j \geq p+1$, one has $|\mathcal{A}_j| = \sum_{i=1}^N |\mathcal{A}_{i,j-p}| \leq N_p \cdot N_{j-p} \leq N_{j-1}^2 = N_j$ (since $j-1 \geq p \geq 1$ and $N_k^2 = N_{k+1}$ for $k \geq 1$). In addition, \mathcal{A}_{j+1} is a refinement of \mathcal{A}_j since each sequence $(\mathcal{A}_{ij})_{j \geq 0}$ is admissible.

Combining the previous bounds and using that $\log N_p = 2^p \log 2 \geq (2^p - 1) \log 2$,

$$\begin{aligned} (2^p - 1) + \sup_{\theta \in A} \sum_{j \geq p} 2^{j/2} \text{diam } A_j(\theta) &\leq (\log 2)^{-1} \log N_p + \max_{1 \leq i \leq N} \sup_{\theta \in A_i} \sum_{j \geq p} 2^{j/2} \text{diam } A_{ij}(\theta) \\ &\leq \max \left\{ \frac{2}{\log 2}, c \right\} \cdot \left[\log N(A, r) + w_A(r) \right], \end{aligned}$$

which proves the upper bound in (4.13).

The second characterization (4.14) in Corollary 4.1 is deduced from the first. Indeed, if $p^*(A) = -\infty$, then $\gamma^*(A) = \gamma_2(A) = \gamma_2^{(0)}(A) < 1$, so that for any $p \geq 1$, $2^p - 1 + \gamma_2^{(p)}(A) \geq 2^p - 1 \geq 1 \geq 2^0 - 1 + \gamma_2^{(0)} = \gamma_2(A)$ and therefore

$$\inf_{p \geq 0} \{(2^p - 1) + \gamma_2^{(p)}(A)\} = \gamma_2(A) = \gamma^*(A).$$

Now, if $p^* = p^*(A) \geq 0$, then (as $\gamma_2^{(p)}(A)$ is non-increasing in p) one has for $p \leq p^*$, $\gamma_2^{(p)}(A) \geq \gamma_2^{(p^*)}(A) \geq 2^{p^*}$, while for $p \geq p^* + 1$ one has $2^p - 1 \geq 2^{p-1} \geq 2^{p^*}$. We deduce that

$$2^{p^*} \leq \inf_{p \geq 0} \{(2^p - 1) + \gamma_2^{(p)}(A)\} \leq (2^{p^*+1} - 1) + \gamma_2^{(p^*+1)}(A) \leq 4 \cdot 2^{p^*}$$

where the last inequality uses that $\gamma_2^{(p^*+1)}(A) \leq 2^{p^*+1}$. Thus, in all cases, one has $\gamma^*(A) \leq \inf_{p \geq 0} \{(2^p - 1) + \gamma_2^{(p)}(A)\} \leq 4\gamma^*(A)$ so that (4.14) follows from (4.13).

A Remaining proofs

A.1 Proof of Proposition 2.1

Proposition 2.1 is a consequence of Theorem 2.1 (recalling also that $V_1(K/\sqrt{2\pi}) = w(K)$), of the Poisson-log-concave inequality (2.17), and of the following lemma.

Lemma A.1. *Let $(v_j)_{j \geq 0} \in \mathbf{R}_+^{\mathbf{N}}$ be a finitely supported sequence such that for every $j \geq 1$,*

$$(j+1)v_{j+1}v_{j-1} \leq jv_j^2. \quad (\text{A.1})$$

Assume also that $v_0 = 1$ and that $v_1 \geq 2$. Then, one has

$$\log \left(\sum_{j \geq 0} v_j \right) \leq 4 \log \left(\max_{j \geq 1} v_j \right) \leq 8 \log \left(\max_{k \geq 0} v_{2^k} \right).$$

Proof. Inequality (A.1) implies that if $n = \inf\{j \geq 1 : v_j = 0\} \in \mathbf{N}$, then $v_j = 0$ for $j \geq n$, and that the sequence $(\alpha_j)_{1 \leq j \leq n}$ defined by $\alpha_j = jv_j/v_{j-1}$ is non-increasing. Also let $\alpha_0 = +\infty$, and define

$$j^* = \max \{j \geq 0 : \alpha_j \geq j\} = \max \{j \geq 0 : v_j \geq v_{j-1}\}.$$

By definition, one has $v_j < v_{j-1}$ for $j > j^*$, and since α_j is non-increasing, $\alpha_j/j = v_j/v_{j-1}$ is decreasing so that for $j \leq j^*$, one has $v_j/v_{j-1} \geq v_{j^*}/v_{j^*-1} \geq 1$, with strict inequality if $j < j^*$. In other words, the sequence $(v_j)_{j \geq 0}$ increases on $\{0, \dots, j^* - 1\}$, reaches its maximum at j^* (possibly also at $j^* - 1$), and decreases on $\{j^*, j^* + 1, \dots\}$.

Now since $\alpha_1 = v_1 \geq 1$, one has $j^* \geq 1$; in addition, $v_1 = \alpha_1 \geq \alpha_{j^*} \geq j^*$, so in particular $j^* \leq v_1 \leq v_{j^*}$ and

$$\sum_{j=0}^{2j^*} v_j \leq (2j^* + 1)v_{j^*} \leq 2v_{j^*}^2 + v_{j^*}.$$

In addition, since (α_j) is non-increasing, for $j \geq 2j^* + 2$ one has $\alpha_j \leq \alpha_{j^*+1} < j^* + 1$ so $v_j/v_{j-1} \leq (j^* + 1)/j \leq 1/2$. Hence for $k \geq 0$, one has $v_{2j^*+1+k} \leq 2^{-k}v_{j^*}$ and thus

$$\sum_{j \geq 2j^*+1} v_j \leq \sum_{k \geq 0} 2^{-k}v_{j^*} \leq 2v_{j^*}.$$

Combining the previous two bounds gives (recalling that $v_{j^*} \geq v_1 \geq 2$)

$$\log \left(\sum_{j \geq 0} v_j \right) \leq \log (2v_{j^*}^2 + 3v_{j^*}) \leq \log(3.5 \cdot v_{j^*}^2) \leq 4 \log v_{j^*}.$$

This corresponds to the first inequality in Lemma A.1. To prove the second one, note that (A.1) implies in particular the log-concavity inequality $v_{j+1}v_{j-1} \leq v_j^2$ for $j \geq 1$, which in turns implies that the sequence $(v_j^{1/j})_{j \geq 1}$ is non-increasing. As a result, let $1 \leq j \leq j^*$ be such that $j^* \leq 2j$. Recalling that $v_{j^*} \geq 1$, one has:

$$v_j^{1/j} \geq v_{j^*}^{1/j^*} \geq v_{j^*}^{1/(2j)},$$

namely $v_{j^*} \leq v_j^2$ i.e. $\log v_{j^*} \leq 2 \log v_j$. Letting $j = 2^k$ where $k \geq 0$ is the maximal integer such that $2^k \leq j^*$ concludes the proof. \square

A.2 Proof of Proposition 3.4

We use the following representation of the Wills functional, which corresponds (up to normalization by $\sqrt{2\pi}$) to (2.12) in the proof of Theorem 2.1 or alternatively [Had75, Eq. (1.4)]:

$$W(A) = 2\pi \int_0^\infty \text{Vol}_n(A + rB_2^n) r e^{-\pi r^2} dr.$$

In particular, for every $t > 0$ one has:

$$\begin{aligned} W(tA)/t^n &= 2\pi \int_0^\infty t^{-n} \text{Vol}_n(tA + rB_2^n) r e^{-\pi r^2} dr \\ &= 2\pi \int_0^\infty \text{Vol}_n\left(A + \frac{r}{t}B_2^n\right) r e^{-\pi r^2} dr. \end{aligned}$$

Now, the family of sets $A + \varepsilon B_2^n$ is increasing (in the sense of inclusion) in $\varepsilon > 0$, with intersection A (since A is closed) and contained for $0 < \varepsilon \leq 1$ in the bounded set $A + B_2^n$, so by dominated convergence one has $\text{Vol}_n(A + \varepsilon B_2^n) \rightarrow \text{Vol}_n(A)$ as $\varepsilon \rightarrow 0$. In particular, for any fixed $r \geq 0$ one has $\text{Vol}_n\left(A + \frac{r}{t}B_2^n\right) r e^{-\pi r^2} \rightarrow \text{Vol}_n(A) r e^{-\pi r^2}$ as $t \rightarrow \infty$, and additionally for every $t \geq 1$ and $r \geq 0$ one has $|\text{Vol}_n\left(A + \frac{r}{t}B_2^n\right) r e^{-\pi r^2}| \leq \kappa_n(\text{diam}(A) + r)^n r e^{-\pi r^2}$ which is integrable over \mathbf{R}^+ , so by dominated convergence, as $t \rightarrow \infty$:

$$W(tA)/t^n \rightarrow 2\pi \int_0^\infty \text{Vol}_n(A) r e^{-\pi r^2} dr = \text{Vol}_n(A).$$

A.3 Simplified form of minimax regret for convex bodies

Theorem 4.2 characterizes the minimax regret in terms of two fixed points for general sets $A \subset \mathbf{R}^n$. We now discuss another characterization in the case where $A = K$ is a convex body.

Since $\mathbf{R}^*(K) \asymp \mathbf{R}^*(K - K)$ by Proposition 5.1, one may assume that K is a symmetric convex body. In this case, Lemma 8.2 ensures that $w_K(r) = w(K \cap rB_2)$ and that $w_K(r)/r$ is decreasing in $r > 0$. Now, one may write:

$$r_*(K) = \sup \left\{ r \geq 0 : w_K(r) \geq r^2 \right\} = \sup \left\{ r \geq 0 : \left(\frac{w_K(r)}{r} \right)^2 \geq r^2 \right\}.$$

Then, using that $w_K(r)/r = w((r^{-1}K) \cap B_2)$ is decreasing in r and Lemma 8.1,

$$\frac{1}{2} \inf_{r > 0} \left\{ w(r^{-1}K \cap B_2) + r^2 \right\} \leq r_*^2(K) \leq \inf_{r > 0} \left\{ w(r^{-1}K \cap B_2)^2 + r^2 \right\}. \quad (\text{A.2})$$

We emphasize that the variational representation (A.2) is specific to the convex case, since for general A the map $r \mapsto w_A(r)/r$ may not be decreasing. Now, for any functions $g, h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$,

$$\frac{1}{2} \left(\inf_{r>0} \{g(r)+r^2\} + \inf_{r>0} \{h(r)+r^2\} \right) \leq \inf_{r>0} \{g(r)+h(r)+r^2\} \leq \inf_{r>0} \{g(r)+r^2\} + \inf_{r>0} \{h(r)+r^2\}.$$

Combining the above leads to the following alternative characterization:

Corollary A.1. *Let $K \subset \mathbf{R}^n$ be an origin-symmetric convex body. Then,*

$$\begin{aligned} \frac{1}{600} \inf_{r>0} \left\{ \log N(K, r) + w(r^{-1}K \cap B_2)^2 + r^2 \right\} &\leq \mathbf{R}^*(K) \\ &\leq 4 \inf_{r>0} \left\{ \log N(K, r) + w(r^{-1}K \cap B_2)^2 + r^2 \right\}. \end{aligned} \tag{A.3}$$

The interest of this representation is that the two “complexity” parameters, namely $\log N(K, r)$ and $w(r^{-1}K \cap B_2)^2$, are both homogeneous to “dimensions” rather than to (powers of) distances. The parameter $w(r^{-1}K \cap B_2)^2 = (w(K \cap rB_2)/r)^2$ has (for $r < \text{diam}(K)/2$) a natural interpretation: it corresponds to the so-called *Dvoretzky-Milman dimension* (see [AGM15, Chapter 5]) of the localized set $K \cap rB_2$.

A.4 Short proof of McMullen’s inequality

In this section, we include a short proof of McMullen’s inequality (3.2) that $\log W(K) \leq V_1(K)$ for any convex body $K \subset \mathbf{R}^n$.

Proof. Let $K \subset \mathbf{R}^n$ be a convex body. One has, as $t \rightarrow 0$,

$$\log W(tK) = \log \left(\sum_{j=0}^n t^j V_j(K) \right) = t \cdot V_1(K) + O(t^2),$$

so that $\log W(tK)/t \rightarrow V_1(K)$ as $t \rightarrow 0$. On the other hand, by concavity of $\log W$ ([AHY21], see Theorem 5.1), the function $t \mapsto \log W(tK)/t$ is non-increasing on \mathbf{R}_+^* . This implies that

$$\log W(K) = \frac{\log W(1 \cdot K)}{1} \leq \lim_{t \rightarrow 0} \frac{\log W(tK)}{t} = V_1(K). \quad \square$$

Acknowledgements. The author warmly thanks Guillaume Lecué for stimulating discussions, Yanjun Han, Yury Polyanskiy, Alexander Rakhlin and Philippe Rigollet for helpful comments, as well as Geoffrey Chinot and Nikita Zhivotovskiy for a careful reading of a first version of this manuscript. This research is supported by a grant of the French National Research Agency (ANR), “Investissements d’Avenir” (LabEx Ecodec/ANR-11-LABX-0047).

References

- [AGM15] S. Artstein-Avidan, A. Giannopoulos, and V. D. Milman. *Asymptotic Geometric Analysis, Part I*. American Mathematical Society, 2015.
- [AHY21] D. Alonso-Gutiérrez, M. A. Hernández Cifre, and J. Yepes Nicolás. Further inequalities for the (generalized) Wills functional. *Commun. Contemp. Math.*, 23(03):2050011, 2021.
- [AKM04] S. Artstein-Avidan, B. Klartag, and V. Milman. The Santaló point of a function, and a functional form of the Santaló inequality. *Mathematika*, 51(1-2):33–48, 2004.
- [ALM⁺14] D. Amelunxen, M. Lotz, M. B. McCoy, and J. A. Tropp. Living on the edge: Phase transitions in convex programs with random data. *Inf. Inference*, 3(3):224–294, 2014.

- [AMM22] H. Aravinda, A. Marsiglietti, and J. Melbourne. Concentration inequalities for ultra log-concave distributions. *Studia Math.*, 265(1):111–120, 2022.
- [Aro50] N. Aronszajn. Theory of reproducing kernels. *Trans. Amer. Math. Soc.*, 68(3):337–404, 1950.
- [Bar87] A. R. Barron. Are Bayes rules consistent in information? In *Open Problems in Communication and Computation*, pages 85–91. Springer, 1987.
- [BBM05] P. L. Bartlett, O. Bousquet, and S. Mendelson. Local Rademacher complexities. *Ann. Statist.*, 33(4):1497–1537, 2005.
- [BFR20] B. Bilodeau, D. Foster, and D. Roy. Tight bounds on minimax regret under logarithmic loss via self-concordance. In *International Conf. on Machine Learning*, pages 919–929, 2020.
- [Bir83] L. Birgé. Approximation dans les espaces métriques et théorie de l’estimation. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 65(2):181–237, 1983.
- [BLM13] S. Boucheron, G. Lugosi, and P. Massart. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.
- [BRY98] A. R. Barron, J. J. Rissanen, and B. Yu. The minimum description length principle in coding and modeling. *IEEE Trans. Inform. Theory*, 44(6):2743–2760, 1998.
- [BvdG11] P. Bühlmann and S. van de Geer. *Statistics for High-Dimensional Data: Methods, Theory and Applications*. Springer-Verlag, 2011.
- [BZ88] Y. D. Burago and V. A. Zalgaller. *Geometric inequalities*. Springer-Verlag, Berlin, 1988.
- [Cat04] O. Catoni. *Statistical Learning Theory and Stochastic Optimization. Ecole d’Eté de Probabilités de Saint-Flour XXXI - 2001*. Lecture Notes in Mathematics. Springer-Verlag, 2004.
- [CBL99] N. Cesa-Bianchi and G. Lugosi. On prediction of individual sequences. *Ann. Statist.*, 27(6):1865–1895, 1999.
- [CBL01] N. Cesa-Bianchi and G. Lugosi. Worst-case bounds for the logarithmic loss of predictors. *Mach. Learn.*, 43(3):247–264, 2001.
- [CBL06] N. Cesa-Bianchi and G. Lugosi. *Prediction, Learning, and Games*. Cambridge University Press, 2006.
- [Cha05] S. Chatterjee. An error bound in the Sudakov-Fernique inequality. *arXiv preprint math/0510424*, 2005.
- [Cha14] S. Chatterjee. A new perspective on least squares under convex constraint. *Ann. Statist.*, 42(6):2340–2381, 2014.
- [Che76] S. Chevet. Processus Gaussiens et volumes mixtes. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 36(1):47–65, 1976.
- [Cov67] T. M. Cover. Behavior of sequential predictors of binary sequences. In *Transactions of the 4th Prague conference on information theory, statistical decision functions, random processes*. Academic Press, 1967.
- [CRP⁺12] V. Chandrasekaran, B. Recht, P. A. Parrilo, and A. S. Willsky. The convex geometry of linear inverse problems. *Found. Comput. Math.*, 12(6):805–849, 2012.
- [CS90] B. Carl and I. Stephani. *Entropy, compactness and the approximation of operators*. Cambridge University Press, 1990.
- [CS04] I. Csiszár and P. C. Shields. Information theory and statistics: A tutorial. *Foundations and Trends in Communications and Information Theory*, 1(4):417–528, 2004.
- [CT06a] E. J. Candès and T. Tao. Near-optimal signal recovery from random projections: Universal encoding strategies? *IEEE Trans. Inform. Theory*, 52(12):5406–5425, 2006.
- [CT06b] T. M. Cover and J. A. Thomas. *Elements of Information Theory*. Wiley Series in Telecommunications and Signal Processing. Wiley-Interscience, 2nd edition, 2006.

- [DJ94] D. L. Donoho and I. M. Johnstone. Minimax risk over l_p -balls for l_q -error. *Probab. Theory Related Fields*, 99(2):277–303, 1994.
- [DMW88] A. De Santis, G. Markowsky, and M. N. Wegman. Learning probabilistic prediction functions. In *Proc. 29th Annual Symposium on Foundations of Computer Science*, pages 110–119, 1988.
- [Don90] D. L. Donoho. Gelfand n -widths and the method of least squares. Technical report, Department of Statistics, University of California Berkeley, 1990.
- [DT09] D. L. Donoho and J. Tanner. Counting faces of randomly projected polytopes when the projection radically lowers dimension. *J. Amer. Math. Soc.*, 22(1):1–53, 2009.
- [Dud67] R. M. Dudley. The sizes of compact subsets of Hilbert space and continuity of Gaussian processes. *J. Funct. Anal.*, 1(3):290–330, 1967.
- [Fer75] X. Fernique. Régularité des trajectoires des fonctions aléatoires gaussiennes. In *Ecole d’Eté de Probabilités de Saint-Flour IV—1974*, pages 1–96. Springer, 1975.
- [FR13] S. Foucart and H. Rauhut. *A Mathematical Introduction to Compressive Sensing*. Birkhäuser, 2013.
- [Gar02] R. Gardner. The Brunn-Minkowski inequality. *Bull. Amer. Math. Soc.*, 39(3):355–405, 2002.
- [Gas18] É. Gassiat. *Universal Coding and Order Identification by Model Selection Methods*. Springer Monographs in Mathematics. Springer, 2018.
- [GLM⁺07] Y. Gordon, A. E. Litvak, S. Mendelson, and A. Pajor. Gaussian averages of interpolated bodies and applications to approximate reconstruction. *J. Approx. Theory*, 149(1):59–73, 2007.
- [GM19] P. D. Grünwald and N. A. Mehta. A tight excess risk bound via a unified PAC-Bayesian-Rademacher-Shtarkov-MDL complexity. In *Proc. 30th International Conference on Algorithmic Learning Theory*, pages 433–465, 2019.
- [Grü07] P. D. Grünwald. *The Minimum Description Length Principle*. MIT Press, 2007.
- [GW93] P. Gritzmann and J. M. Wills. Lattice points. In *Handbook of convex geometry. Volume B*, pages 765–797. Amsterdam: North-Holland, 1993.
- [Had75] H. Hadwiger. Das Will’sche funktional. *Monatsh. Math.*, 79(3):213–221, 1975.
- [HO97] D. Haussler and M. Opper. Mutual information, metric entropy and cumulative relative entropy risk. *Ann. Statist.*, 25(6):2451–2492, 1997.
- [IH81] I. Ibragimov and R. Has’minskii. *Statistical estimation: asymptotic theory*. Springer, 1981.
- [Kah86] J.-P. Kahane. Une inégalité du type de Slepian et Gordon sur les processus gaussiens. *Israel J. Math.*, 55(1):109–110, 1986.
- [KL51] S. Kullback and R. A. Leibler. On information and sufficiency. *Ann. Math. Stat.*, 22(1):79–86, 1951.
- [KM05] B. Klartag and V. D. Milman. Geometry of log-concave functions and measures. *Geom. Dedicata*, 112(1):169–182, 2005.
- [Kol11] V. Koltchinskii. *Oracle Inequalities in Empirical Risk Minimization and Sparse Recovery Problems*, volume 2033 of *École d’Été de Probabilités de Saint-Flour*. Springer-Verlag, 2011.
- [KR97] D. A. Klain and G.-C. Rota. *Introduction to Geometric Probability*. Cambridge University Press, 1997.
- [LC73] L. Le Cam. Convergence of estimates under dimensionality restrictions. *Ann. Statist.*, 1(1):38–53, 1973.
- [LCY00] L. Le Cam and G. L. Yang. *Asymptotics in statistics: some basic concepts*. Springer, 2000.
- [LM13] G. Lecué and S. Mendelson. Learning subgaussian classes: Upper and minimax bounds. Technical report, CNRS, Ecole polytechnique and Technion, 2013.

- [LMN⁺20] M. Lotz, M. B. McCoy, I. Nourdin, G. Peccati, and J. A. Tropp. Concentration of the intrinsic volumes of a convex body. In B. Klartag and E. Milman, editors, *Geometric Aspects of Functional Analysis*, Lecture Notes in Mathematics, pages 139–167, 2020.
- [LW94] N. Littlestone and M. K. Warmuth. The weighted majority algorithm. *Inform. and Comput.*, 108(2):212–261, 1994.
- [Mas00] P. Massart. Some applications of concentration inequalities to statistics. *Ann. Fac. Sci. Toulouse Math.*, 9(2):245–303, 2000.
- [McM91] P. McMullen. Inequalities between intrinsic volumes. *Monatsh. Math.*, 111(1):47–53, 1991.
- [Men03] S. Mendelson. On the performance of kernel classes. *J. Mach. Learn. Res.*, 4:759–771, 2003.
- [Men17] S. Mendelson. “Local” vs. “global” parameters—breaking the Gaussian complexity barrier. *Ann. Statist.*, 45(5):1835–1862, 2017.
- [MF98] N. Merhav and M. Feder. Universal prediction. *IEEE Trans. Inform. Theory*, 44:2124–2147, 1998.
- [Mou23] J. Mourtada. A short proof of concentration of the intrinsic volume sequence of a convex body. *In preparation*, 2023.
- [MY19] K. Miyaguchi and K. Yamanishi. Adaptive minimax regret against smooth logarithmic losses over high-dimensional ℓ_1 -balls via envelope complexity. In *Proc. 22nd conference on Artificial Intelligence and Statistics*, pages 3440–3448, 2019.
- [Ney22] M. Neykov. On the minimax rate of the Gaussian sequence model under bounded convex constraints. *Preprint arXiv:2201.07329*, 2022.
- [OH99] M. Opper and D. Haussler. Worst case prediction over sequences under log loss. In *The mathematics of information coding, extraction and distribution*, pages 81–90. Springer, 1999.
- [Pis99] G. Pisier. *The volume of convex bodies and Banach space geometry*. Cambridge University Press, 1999.
- [PW22] Y. Polyanskiy and Y. Wu. *Information theory: From coding to learning*. Book draft; forthcoming in Cambridge University Press, 2022+.
- [Ris85] J. J. Rissanen. *Minimum description length principle*. Wiley Online Library, 1985.
- [Ris96] J. J. Rissanen. Fisher information and stochastic complexity. *IEEE Trans. Inform. Theory*, 42(1):40–47, 1996.
- [RS57] C. A. Rogers and G. C. Shephard. The difference body of a convex body. *Arch. Math.*, 8(3):220–233, 1957.
- [RS14] A. Rakhlin and K. Sridharan. Online non-parametric regression. In *Proc. 27th Conference on Learning Theory*, pages 1232–1264. PMLR, 2014.
- [RS15] A. Rakhlin and K. Sridharan. Sequential probability assignment with binary alphabets and large classes of experts. *Preprint arXiv:1501.07340*, 2015.
- [RV08] M. Rudelson and R. Vershynin. On sparse reconstruction from Fourier and Gaussian measurements. *Comm. Pure Appl. Math.*, 61(8):1025–1045, 2008.
- [SC08] I. Steinwart and A. Christmann. *Support Vector Machines*. Springer-Verlag, 2008.
- [Sch84] C. Schütt. Entropy numbers of diagonal operators between symmetric Banach spaces. *J. Approx. Theory*, 40(2):121–128, 1984.
- [Sch13] R. Schneider. *Convex bodies: The Brunn-Minkowski theory*. Cambridge University Press, second edition, 2013.
- [Sht87] Y. M. Shtarkov. Universal sequential coding of single messages. *Probl. Inf. Transm.*, 23(3):3–17, 1987.

- [SKF08] M. W. Seeger, S. M. Kakade, and D. P. Foster. Information consistency of nonparametric Gaussian process methods. *IEEE Trans. Inform. Theory*, 54(5):2376–2382, 2008.
- [Sle62] D. Slepian. The one-sided barrier problem for Gaussian noise. *Bell System Technical Journal*, 41(2):463–501, 1962.
- [Sto09] M. Stojnic. Various thresholds for ℓ_1 -optimization in compressed sensing. *Preprint arXiv:0907.3666*, 2009.
- [Sud76] V. N. Sudakov. Geometric problems of the theory of infinite-dimensional probability distributions. *Trudy Mat. Inst. Steklov*, 141:3–191, 1976.
- [SW08] R. Schneider and W. Weil. *Stochastic and Integral Geometry*. Springer, 2008.
- [Tal87] M. Talagrand. Regularity of Gaussian processes. *Acta Math.*, 159:99–149, 1987.
- [Tal11] M. Talagrand. *Mean Field Models for Spin Glasses. Volume I: Basic Examples*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 2011.
- [Tal21] M. Talagrand. *Upper and lower bounds for stochastic processes: Decomposition Theorems*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer, 2nd edition, 2021.
- [Tsi82] B. S. Tsirel’son. A geometrical approach to maximum likelihood estimation for infinite-dimensional Gaussian location. I. *Theory Probab. Appl.*, 27(2):411–418, 1982.
- [Tsi86] B. S. Tsirel’son. A geometric approach to maximum likelihood estimation for infinite-dimensional Gaussian location. II. *Theory Probab. Appl.*, 30(4):820–828, 1986.
- [Tsy09] A. B. Tsybakov. *Introduction to nonparametric estimation*. Springer, 2009.
- [vdG99] S. van de Geer. *Empirical Processes in M-estimation*. Cambridge University Press, 1999.
- [Ver18] R. Vershynin. *High-dimensional probability: An introduction with applications in data science*. Cambridge University Press, 2018.
- [Vit96] R. A. Vitale. The Wills functional and Gaussian processes. *Ann. Probab.*, 24(4):2172–2178, 1996.
- [Vov98] V. Vovk. A game of prediction with expert advice. *J. Comput. System Sci.*, 56(2):153–173, 1998.
- [Vov01] V. Vovk. Competitive on-line statistics. *Int. Stat. Rev.*, 69(2):213–248, 2001.
- [Wai19] M. J. Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2019.
- [Wil73] J. M. Wills. Zur gitterpunktanzahl konvexer mengen. *Elem. Math.*, 28:57–63, 1973.
- [Yan00] Y. Yang. Mixing strategies for density estimation. *Ann. Statist.*, 28(1):75–87, 2000.
- [YB99] Y. Yang and A. R. Barron. Information-theoretic determination of minimax rates of convergence. *Ann. Statist.*, 27(5):1564–1599, 1999.